This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

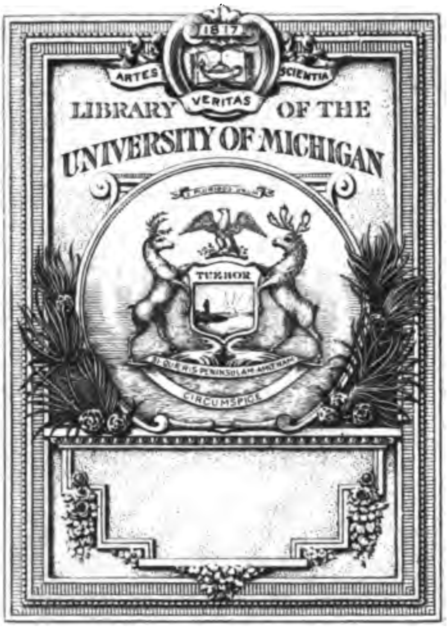
+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`


1817
ARTES
VERITAS
SCIENTIA
LIBRARY OF THE
UNIVERSITY OF MICHIGAN
E PLURIBUS UNUM
TUEBOR
SI QUAERIS PENINSULAM AMOENAM
CIRCUMSPICE

QA
41
B87

A SYNOPSIS

OF THE PRINCIPAL

FORMULÆ AND RESULTS

OF

Pure Mathematics.

BY

CHARLES BROOKE, M.B.

OF ST. JOHN'S COLLEGE.

CAMBRIDGE:
Printed by J. Smith, Printer to the University.
SOLD BY J. & J. J. DEIGHTON, CAMBRIDGE;
AND C. J. G. & F. RIVINGTON, LONDON.

1829

History of Science
Wheldon
10-13-31
24463

PREFACE.

The Author of the following pages, having experienced the want of a compendious collection of results, in common probably with most mathematical students, endeavoured, during his undergraduateship, to form such a collection for his own use. This has since that time been arranged, and considerably enlarged, and is now laid before the Public, in the hope that it may in some measure facilitate the labour of the analyst, by enabling him to compare and apply the results of previous investigation.

C3.26-347 M6

It was originally intended to restrict these pages to a mere collection of formulæ, but it is hoped that in adding an outline of many useful methods of operation, sufficiently detailed to render them applicable to particular cases, the increased size of the work will not be deemed an objection.

The references that have been given will probably be found sufficiently minute to enable the reader to find without difficulty any demonstration he may require; had the proof of each result been separately quoted, the size of the work must unavoidably have been much increased. The object has been to refer to works of the greatest authority, and to those which can most readily be procured.

The principles of Notation that have been adopted in the following pages are explained in the Appendix; where will also be found an explanation of the very ingenious and powerful symbols invented by Mr. Jarrett, of Catharine Hall; to whom the Author is indebted for several of the theorems expressed by those symbols, that have been introduced in the notes.

The great difficulty of printing mathematical works correctly, may be fairly estimated by the number of errors that occur in the works of the ablest writers on the subject. As it is probable that some few may still have escaped the Author's notice, he will feel much indebted to any of his readers, who will have the kindness to communicate to the publishers any errors they may happen to detect.

The Author begs leave to take this opportunity of expressing his acknowledgements to the Syndics of the University Press, for the very liberal manner in which they have contributed to defray the expenses of the work.

St. John's College,
May, 1829.

◆

TABLE OF CONTENTS.

◆

ALGEBRA.

ART.		PAGE
1—3.	Definitions and rules	1
4—6.	Powers and roots	*ib.*
7.	Surds	2
8, 9	Greatest common divisor	3
10—12.	Fractions	*ib.*
13—15.	Equations, simple and quadratic	4
16, 17.	Ratios and proportion	7
18, 19.	Arithmetical progression	8
20.	Harmonical progression	10
21.	Figurate and polygonal numbers	*ib.*
22, 23.	Geometrical progression	11
24.	Permutations and combinations	13
25.	Binomial theorem	14
26, 27.	Polynomial theorem	17
28.	Indeterminate coefficients	22
29.	Logarithms	*ib.*
30, 31.	Continued fractions	28
32.	General properties of equations	36
33.	Transformation of equations	38
34.	Elimination	40
35, 36.	Depression of equations	43
37, 38.	Symmetrical functions of the roots of equations	44
39.	The equation of differences	47
40.	Limits of the roots of equations	48
41.	Commensurable roots	50
42.	Incommensurable roots	51
43.	Quadratic factors	53
44.	Impossible roots	*ib.*
45.	Application of the theory of equations to surds	54

ART. PAGE
46. Cubic equations 55
47. Biquadratic equations 56
48—50. The equation $y^n \pm p=0$ 59
51. General solution of equations 62
52—56. Indeterminate equations of the first degree 63
57. Forms of square numbers 73
58—64. Indeterminate equations of the second degree 77
65. Forms of cubes 86
66. Indeterminate equations of the third degree 87
67. Forms of biquadrates 88
68. Indeterminate equations of the fourth degree 89
69. Solution of homogeneous indeterminate equations 90
70. Solution of the equation $x^n-b=ay$ 91
71. General properties of numbers 92
72. Properties of prime numbers 93
73. Quadratic forms of prime numbers 94
74. Resolution of numbers into squares 95
75. Quadratic divisors 96
76. Ternary divisors 97
77. Scales of notation 99

TRIGONOMETRY.

1, 2. Divisions of the circle 100
3—6. Relations of the trigonometrical lines 101
7—9. Values of sin *a*, cos *a*, and tan *a* 102
10. Formulæ relating to two arcs 104
11. Formulæ relating to double arcs 105
12. Values of the sine, cosine, &c. of 30°, 45°, and 60° 106
13. Formulæ relating to three arcs 107
14—18. Relations between the sides and angles of plane triangles *ib.*
19. Solution of right-angled plane triangles 109
20. Solution of oblique-angled plane triangles 110
21. General principles of spherical trigonometry 112
22. Relations between the sides and angles of spherical triangles 113
23. Formulæ for the area of a spherical triangle 116
24. Values of the radii of the inscribed and circumscribing circles 117

Art.		Page
25.	Solution of right-angled spherical triangles	117
26.	Solution of oblique-angled spherical triangles	121
27.	Series for the sine, cosine, &c. in terms of the arc	128
28.	Series expressing the inverse circular functions	129
29.	Series for determining the value of π	130
30.	Formulæ involving impossible quantities	ib.
31—33.	Formulæ for the sums of arcs, and multiple arcs	131
34.	Powers of the sine and cosine of an arc	136
35.	Sums of trigonometrical series	137
36.	Resolution of trigonometrical quantities into factors	140
37.	Approximate solution of triangles	141
38.	Solution of triangles by series	142
39—41.	Formulæ peculiar to geodetic operations	143
42—48.	Formulæ for the construction of tables	144
49.	Formulæ for the verification of tables	150
50—53.	Trigonometrical solution of equations	151
54.	Properties of a quadrilateral inscribed in a circle	153
55, 56.	Properties of polygons	154

Analytical Geometry.

1—3	Definitions and principles	155
	Analytical Geometry of two dimensions.	
4, 5.	The equation of a straight line, and its properties	155
6, 7.	Transformation of co-ordinates	157
8—10.	The circle	158
11—13.	The parabola	160
14—16.	The ellipse	163
17—20.	The hyperbola	167
21, 22.	Discussion of lines of the second order	172
23, 24	Summary of equations	174
	Analytical Geometry of three dimensions.	
25.	The straight line	176
26, 27.	The plane	177
28.	The orthogonal projection of plane figures	179
29.	Oblique co-ordinates	180
30—33.	Transformation of co-ordinates	181
34.	The sphere	183

ART. PAGE
35. The cylinder 184
36, 37. The cone . . . . ib.
38—42. Surfaces of the second order 185
43—46. The intersection of a surface of the second order, and a plane 189
47. The tangent plane 191

DIFFERENTIAL CALCULUS.

1. Differentiation of algebraic functions 193
2. Differentiation of exponential functions 194
3—6. Successive differentiation . . . . ib.
7. Differentiation of functions of superior orders . . . 196
8—10. Development of functions 197
11. Implicit functions 199
12. Transformation of the independent variable . . . . ib.
13. Elimination of an arbitrary function 200
14, 15. Particular values of the variable . . . . ib.
16. Maxima and minima 203

INTEGRAL CALCULUS.

1. Fundamental formulæ 204
2. Inverse circular functions . . . . ib.
3. Logarithmic integrals 205
4, 5. Decomposition of rational fractions . . . . . . ib.
6. Formulæ for rendering surds rational 210
7—15. Integration of irrational functions 211
16, 17. Integration of elliptic transcendents 218
18. Integration of exponential and logarithmic functions 222
19—22. Integration of circular functions 224
23. Approximate values of integrals 228
24, 25. Successive integration 230
26. Integration of functions of several variables . . . ib.

Differential Equations.

27—32. Differential equations of the first order, and of one dimension in $d_x y$ 231
33—34. The introduction of a factor which renders a differential equation integrable 233
35. Singular solutions 235

ART. PAGE

36, 37. Equations of more than one dimension in $d_x y$ . . . 236

38—42. Differential equations of the second order 239

43—45. Linear equations of the n^{th} order 240

46—48. Simultaneous equations 245

49—52. Approximate integration of differential equations . . 248

53, 54. Comparison of elliptic transcendents 249

55. Total differential equations of several variables . . . 251

56—61. Partial differential equations of the first order . . . 252

62—67. Partial differential equations of superior orders . . . 255

68. Integration of partial differential equations by series . 261

69. Singular solutions of partial differential equations . . *ib.*

APPLICATION OF THE DIFFERENTIAL AND INTEGRAL CALCULUS TO GEOMETRY.

1—3. The contact of lines 263

4, 5. Asymptotes 265

6—8. Singular points 266

9—11. Curves referred to polar co-ordinates 267

12, 13. Circumscribing figures 268

14, 15. The contact of surfaces 269

16, 17. The contact of curves of double curvature 271

18—23. Rectification 272

24—26. Quadrature 274

27. Cubature 275

28. Transformation of co-ordinates . . . . *ib.*

29. Conditions which render a curve quadrable 276

30. Conditions which render a curve rectifiable . . . . *ib.*

31. Trajectories . . . . *ib.*

32—38. Remarkable algebraical curves 277

39—48. Remarkable transcendental curves 279

CALCULUS OF VARIATIONS 288

CALCULUS OF FINITE DIFFERENCES.

Direct method of differences.

1. Fundamental formulæ 292

2, 3. Successive differences . . . . *ib.*

4. Series involving the differences and differential coefficients 294

ART.		PAGE
5.	Differences of functions of two or more variables	295
6—9.	Interpolation of series	ib.
10.	Differences of the trigonometrical lines	298
11—14.	The variation of triangles	299
14—18.	The construction of logarithmic and trigonometrical tables	305

Inverse method of differences.

19—20.	Integration of algebraic functions	309
21.	The numbers of BERNOULLI	310
22.	Integration of exponential functions	311
23, 24.	Successive integration	312
25—32.	Equations of differences	314
33, 34.	Equations of mixed differences	319
35—39.	Summation of series	320
40—42.	Theory of generating functions	326

FUNCTIONAL EQUATIONS.

1—3.	Reduction of functional equations to equations of differences	330
4—9.	General solution of functional equations obtained from a particular solution	332
10, 11.	Differential functional equations	339
	APPENDIX	341

The following symbols, although not original, may perhaps require explanation, as they have not yet been generally introduced.

$a \ngtr b$ is read *a* is not greater than *b*,

$a \nless b$ . . . . *a* is not less than *b*,

$a \mathrel{∻} b$ . . . . *a* is divisible by *b*,

$a \mathrel{⊹} b$ . . . . *a* is of the same form as *b*.

LIST OF WORKS

WHICH HAVE BEEN REFERRED TO BY INITIAL LETTERS.

W.	Wood, Algebra.
E.	Euler, Algebra.
E. Add.	Additions in the second edition of Euler's Algebra.
F.	Francœur, Cours complète de Mathematiques pures.
Bour.	Bourdon, Algèbre.
L.	Lacroix, Eléments d' Algèbre.
L. C.	Lacroix, Compléments d' Algèbre.
B.	Barlow, Theory of Numbers.
Leg.	Legendre, Théorie des Nombres.
G.	Garnier, Eléments d' Algèbre.
G. *A.*	Garnier, Analyse Algèbrique.
E. M.	Encyclopedia Metropolitana.
Leg.	(In Trigonometry.) Legendre, Trigonométrie.
W.	- - - - - - - - - - Woodhouse, Trigonometry.
L.	- - - - - - - - - - Lardner, Trigonometry.
C.	Cagnoli, Trigonométrie.
Biot,	Biot, Essai de Géométrie Analytique.
G. G. *A.*	Garnier, Géométrie Analytique.
H. A. G.	Hamilton, Analytical Geometry.
H. C. S.	Hamilton, Conic Sections.
L. A. G.	Lardner, Algebraic Geometry.
Hust.	Hustler, Conic Sections.
L. C. D.	Lacroix, Traitè du Calcul Differentiel, &c.
L. D. C.	Lardner, Differential Calculus.
Tr. *L.*	Translation of Lacroix, Differential Calculus.
L. Mec. Cel.	Laplace, Mécanique Céleste.

ERRATA.

PAGE	*LINE*				
5	12	*for*	numerator	*read*	denominator
23	5	...	Log	....	log
32	8	...	a	....	$\frac{a}{b}$
36	15	...	$-a_n$	....	$-\alpha_n$
52	6	...	$>$	....	$<$
98	14	*after*	same	*insert*	time
119	*two last*	*for*	$\cos B$	*read*	$\sin B$
131	13	...	sin	....	cos
	14	...	cos	....	sin
154	15	...	circumscribed	....	inscribed
181	9	...	$\frac{B_1 C_2 + B_2 C_1}{ef} \cos \gamma$	....	$\frac{B_1 C_2 + B_2 C_1}{ef} \cos \alpha$
195	13	...	$(\sec x^2)$	....	$(\sec x)^2$
	21	...	$(1-x)^2$	....	$(1-x^2)$
196	1, *Note*	...	m in the denominator	....	$\underline{\lvert m}$
198	5, 6, *Note*	...	$d_{x=0}^{n-1}:u$	....	$d_{y=0}^{n-1}:u$
199	5	*after*	then	*insert*	, if $u=0$,
203	11	....	does	.....	not
210	5	*for*	$a^1 + b^1 x^n$	*read*	$a_1 + b_1 x^n$
230	14	...	functions	....	constants
231	3	...	v_1	....	$v,$
232	6 *from the bottom*	...	c_2	....	$c_2 y$
244	4 . :	...	$d_r s = s$	....	$d_t s = s d_t r$
267	*last*	...	$\frac{rp}{(r^2 - p^2)^{\frac{1}{2}}}$	....	$\frac{up}{(u^2 - p^2)^{\frac{1}{2}}}$
274	6 *from the bottom*	...	$(\int_{\theta=\beta}$	....	$\frac{1}{2}(\int_{\theta=\beta}$
288	4	...	$\int\{$	....	$\int_x \{$
346	*last*	...	$d_y^u u$	....	$d_y^n u.$

ALGEBRA.

Art. (1.) Definitions and Axioms. (*W.* 47—72; &c.)

(2.) ***Rules for addition, and subtraction.***

$a + (b - c) = a + b - c.$

$a - (b - c) = a - b + c.$

$a + (-c) = a - c.$

$a - (-c) = a + c.$

(3.) ***Rules for multiplication and division.***

$(a - b)c = ac - bc.$

$(a - b)(c - d) = ac - bc - ad + bd.$

$(a - b)(-d) = -ad + bd.$

$(-b)(-d) = bd.$ (*F.* 94—7.)

The general rule for the signs both in multiplication, and division is, that like signs give a positive, and unlike, a negative result.

(4.) $a^n = a.a.a \ldots\ldots$ to n factors.

$a^m \times a^n = a^{m+n}.$ (1)

$(a^m)^n = a^{mn}.$

If n is not a positive integer, the signification of a^n is conventional, being merely such as to satisfy equation (1).

$a^{m-n} = a^m \div a^n.$

$a^0 = a^{m-m} = 1.$

$a^{-n} = a^{0-n} = \dfrac{1}{a^n}.$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (a^m)^{\frac{1}{n}} = (a^{\frac{1}{n}})^m.$$

$$(ab)^m = a^m . b^m.$$

$$(ab)^{\frac{m}{n}} = a^{\frac{m}{n}} . b^{\frac{m}{n}}.$$

(*W.* 78—110; *E.* 180—219; *F.* 131—4.)

(5.) $(-1)^{2m} = 1$, and $(-1)^{2m-1} = -1$; m being any integer.

Nature of impossible quantities. (*E.* 139—51.)

$$(+a)^{\frac{1}{2m}} = \pm a^{\frac{1}{2m}}.$$

$$(-a)^{\frac{1}{2m}} = \pm a^{\frac{1}{2m}}.(-1)^{\frac{1}{2m}}.$$

$a^{\frac{1}{2m-1}}$ has the same sign as a.

(6.) Nature of squares, and square roots. (*E.* 123—38.)

Nature of cubes, and cube roots. (*E.* 152—67.)

Extraction of the square and cube roots of numerical, and algebraical quantities. (*W.* 126—33; *E.* 317—39.)

SURDS.

(7.) Properties of surds. (*W.* 239—51; *B.* 18, 19.)

The square root of a quantity cannot be partly rational, and partly a quadratic surd.

If the equation $x + \sqrt{y} = a + \sqrt{b}$ is generally true, then

$$x = a, \quad \text{and} \quad y = b.$$

The square root of an integer that is not a complete square can neither be expressed by an integer, nor by a rational fraction.

Neither the sum nor difference of the square roots of two quantities prime to each other can be represented by any rational quantity, nor by the square root of any rational quantity, unless each be a complete square.

The product of two quadratic surds, which cannot be reduced to others having the same irrational part, is irrational.

To extract the square root of $a+\sqrt{b}$:

Assume $$a+\sqrt{b}=\sqrt{x}+\sqrt{y};$$
then $$x=\tfrac{1}{2}\{a+(a^2-b)^{\frac{1}{2}}\},$$
$$y=\tfrac{1}{2}\{a-(a^2-b)^{\frac{1}{2}}\}.$$

x and y are rational only when a^2-b is a complete square.

(*W.* 258; *Bour.* 118—21.)

Method of extracting the n^{th} root of a binomial quadratic surd. (*W.* 259.)

GREATEST COMMON DIVISOR.

(8.) The greatest common divisor of two quantities may be found by arranging both according to the powers of some one letter; then dividing the greater by the less, and the preceding divisor always by the last remainder, until the remainder is equal to nothing; the last divisor will be the greatest common divisor required.

If any divisor contains a factor not contained in both the given quantities, that factor may be rejected.

(*Bour.* 34—41, 258—74; *W.* 90; *E.* 451—60.)

(9.) The least common multiple of two quantities is their product divided by their greatest common divisor.

To find the least common multiple of n quantities: find the least common multiple of the first and second; then of the quantity thus obtained and the third, and so on: the result will be the least common multiple required. (*W.* 377—9.)

FRACTIONS.

(10.) Nature of fractions. (*E.* 68—93.)

(11.) *Rules for the multiplication and division of fractions.*

$$\frac{a}{b}=\frac{-a}{-b}.$$

$$\frac{-a}{b}=\frac{a}{-b}.$$

$$\frac{a\times c}{b\times c}=\frac{a}{b}.$$

$$\frac{a \div c}{b \div c} = \frac{a}{b}.$$

$$\frac{a}{b} \times c = \frac{ac}{b} = \frac{a}{b \div c}.$$

$$\frac{a}{b} \div d = \frac{a}{bd} = \frac{a \div d}{b}.$$

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}.$$

$$\frac{a}{d} + \frac{b}{d} - \frac{c}{d} = \frac{a+b-c}{d}.$$ (*W.* 98—108.)

(12.) To reduce a fraction to its lowest terms: divide both the numerator and denominator by their greatest common divisor.

To reduce fractions to a common denominator: multiply each numerator by all the denominators except its own, for a new numerator; and all the denominators together, for a new denominator. (*W.* 96.)

If the least common denominator be required; find the least common multiple of all the denominators, for a new denominator; and multiply each numerator by the quotient of the common denominator divided by the corresponding denominator, for a new numerator. (*Bonnycastle, Arith.*)

Equations.

(13.) *General rules for the solution of equations.*

The same quantity may be added to, or subtracted from, both sides of an equation.

Any quantity may be transposed from one side of an equation to the other, its sign being changed.

The signs of all the terms may be changed.

Every term may be multiplied, or divided, by the same quantity.

Both sides may be raised to the same power.

The same root of both sides may be taken.

(*W.* 134—41; *Bour.* 43—5; &c.)

(14.) *Simple Equations.* Every simple equation containing one unknown quantity, when cleared of fractions, and surds, may be reduced to the form

$$ax = b;$$

whence $x = \frac{b}{a}$. (*W.* 134—42; *E.* 573—604.)

To clear an equation of fractions: find the least common multiple of all the denominators; and multiply each numerator by the quotient of that quantity divided by the corresponding numerator.

Two simple equations containing two unknown quantities may be reduced to the form

$$a_1 x + b_1 y = c_1; \qquad (1)$$

$$a_2 x + b_2 y = c_2. \qquad (2)$$

These two equations may be reduced to one, containing a single unknown quantity, by either of the following methods:

1st. from (1), $x = \frac{c_1 - b_1 y}{a_1}$,

and substituting this value in (2), $\frac{a_2}{a_1} . \overline{c_1 - b_1 y} = c_2$.

2nd. from (1), $x = \frac{c_1 - b_1 y}{a_1}$;

(2), $x = \frac{c_2 - b_2 y}{a_2}$;

$$\therefore \frac{c_1 - b_1 y}{a_1} = \frac{c_2 - b_2 y}{a_2}.$$

3rd. if $a_1 = ne_1$, $a_2 = ne_2$, then

(1) × e_2 — (2) × e_1, $(e_2 b_1 - e_1 b_2)\cdot y = e_2 c_1 - e_1 c_2$.

(*W.* 143; *E.* 605—12.)

The same principles may easily be extended to three or more independent equations containing as many unknown quantities. (*E.* 613—22.)

General method of deducing the value of n unknown quantities from n simple equations.

(*Schweins, Theor. der Diff.* 183; *F.* 111.)

(15.) *Quadratic Equations.* Every quadratic equation may, by clearing it of fractions and surds, and dividing every term by the coefficient of x^2, be reduced to the form

$$x^2 + px = q;$$

adding to each side the square of $\frac{1}{2}$ the coefficient of the second term, we have

$$\overline{x + \frac{p}{2}}\Big|^{2} = \frac{p^2}{4} + q,$$

whence $x = \frac{1}{2}\{-p \pm (p^2 + 4q)^{\frac{1}{2}}\}$.

The following are some of the principal forms of equations which may be solved as quadratics:

$$x^{2n} + px^n = q.$$

$$\overline{\frac{a_1 x^2 + b_1 x + c_1}{a_2 x^2 + b_2 x + c_2}}\Bigg|^{2} + p\cdot\frac{a_1 x^2 + b_1 x + c_1}{a_2 x^2 + b_2 x + c_2} = q.$$

$$\overline{ax^2 + bx + c}\,\big|^{2} + p\,.\,\overline{ax^2 + bx + c} = x^2 \pm px.$$

$$\overline{\frac{a_1 x + b_1}{a_2 x + b_2}}\Bigg|^{2} + p\cdot\frac{a_1 x + b_1}{a_2 x + b_2} = x^2 \pm px.$$

All equations containing only one power of the unknown quantity may be solved as simple equations: having found

$x^n = a$, we have $x = a^{\frac{1}{n}}$. (*W.* 147—50; *E.* 638—55.)

Various artifices applicable to particular occasions.

(*W.* 151—6; *Bland, Alg. Prob.*)

RATIOS, AND PROPORTION.

(16.) *Ratios.* Principles. (*E.* 440—50.)

If $a > b$, then

$a : c > b : c$;

$c : a < c : b$.

$a + x : b + x >$ or $< a : b$, according as $a <$ or $> b$.

(*W.* 157—70.)

(17.) *Proportion.* If $\frac{a}{b} = \frac{c}{d}$, then

$a : b :: c : d$;

$ad = bc$;

$b : a :: d : c$;

$a : c :: b : d$;

$a \pm b : b :: c \pm d : d$;

$a - c : b - d :: c : d$;

$a : a - b :: c : c - d$;

$a + b : a - b :: c + d : c - d$;

and generally, $ma + nb : pa + qb :: mc + nd : pc + qd$.

(*W.* 171—182; *E.* 461—5.)

If $a : b :: c : d :: e : f ::$ &c. then

$a : b :: a + c + e +$ &c $: b + d + f +$ &c.

$ma : mb :: \frac{c}{n} : \frac{d}{n}$;

$ma : \frac{b}{n} :: mc : \frac{d}{n}$;

$a^n : b^n :: c^n : d^n$.

If $a : b :: c : d$,

and $e : f :: g : h$; then

$ae : bf :: cg : dh$. (*W.* 184—8.)

If two numbers be prime to each other they are the least in that proportion. (*W.* 189.)

Compound proportion. (*E.* 488—504.)

ARITHMETICAL PROGRESSION.

(18.) Let a_m represent the m^{th} term of an arithmetical series,

b, the common difference,

s, the sum of n terms;

$$a_m = a_1 + (m-1)\,b.$$

$$s = \frac{n}{2}\,(2a_1 + \overline{n-1}\,.\,b.)$$

$$a_{n-m} + a_{n+m} = 2a_n.$$

To insert n arithmetical means between c, and e.

$$\text{The } m^{\text{th}} \text{ mean} = c + \frac{m}{n+1}\,(e-c).$$

Given the p^{th} and q^{th} terms, to find the r^{th} term.

$$a_r = a_p + \frac{r-p}{q-p}\,(a_q - a_p.)$$

(*W.* 212; *E.* 402—24; *Bour.* 193—8.)

(19.) Any three of the quantities a_1, a_n, b, n, s; being given, the others may be found from the following formulæ:

$$(a_1,\ b,\ n);\ \begin{cases} a_n = a_1 + \overline{n-1}\,.\,b. \\ s = \dfrac{n}{2}\,\{2a_1 + \overline{n-1}\,.\,b\}. \end{cases}$$

$$(a_1,\ a_n,\ b);\ \left\{\begin{aligned} n &= 1 + \frac{a_n - a_1}{b}. \\ s &= \frac{a_n + a_1}{2} + \frac{\overline{a_n + a_1} \,.\, \overline{a_n - a_1}}{2b}. \end{aligned}\right.$$

$$(a_1,\ b,\ s);\ \left\{\begin{aligned} n &= \frac{b - 2a_1}{2b} \pm \left(\frac{2s}{b} + \overline{\frac{2a_1 - b}{2b}}\Big|^2\right)^{\frac{1}{2}}. \\ a_n &= \frac{-b}{2} \pm \left\{2bs + \overline{a_1 - \frac{b}{2}}\Big|^2\right\}^{\frac{1}{2}}. \end{aligned}\right.$$

$$(a_n,\ b,\ n);\ \left\{\begin{aligned} a_1 &= a_n - \overline{n-1} \,.\, b. \\ s &= \frac{n}{2}(2a_n - \overline{n-1} \,.\, b). \end{aligned}\right.$$

$$(b,\ n,\ s);\ \left\{\begin{aligned} a_1 &= \frac{s}{n} - \frac{n-1}{2}\, b. \\ a_n &= \frac{s}{n} + \frac{n-1}{2}\, b. \end{aligned}\right.$$

$$(a_1,\ a_n,\ n);\ \left\{\begin{aligned} b &= \frac{a_n - a_1}{n-1}. \\ s &= \frac{n}{2}(a_1 + a_n). \end{aligned}\right.$$

$$(a_1,\ n,\ s);\ \left\{\begin{aligned} b &= 2\,\frac{s - a_1 n}{n(n-1)}, \\ a_n &= \frac{2s}{n} - a_1. \end{aligned}\right.$$

$$(a_1,\ a_n,\ s);\ \left\{\begin{aligned} b &= \frac{\overline{a_n + a_1} \,.\, \overline{a_n - a_1}}{2s - a_1 - a_n}. \\ n &= \frac{2s}{a_1 + a_n}. \end{aligned}\right.$$

The value of a circulating decimal is a fraction whose numerator is the period, and denominator a number consisting of as many *nines* as there are digits in the period.

(*W.* 225; *F.* 50—2.)

(23.) Any three of the five quantities a_1, a_n, r, n, s, being given, the other two may be found from the following formulæ.

$$(a_1,\ r,\ n);\quad \begin{cases} a_n = a_1 r^{n-1}. \\ s = a_1 \dfrac{r^n - 1}{r - 1}. \end{cases}$$

$$(a_1,\ a_n,\ n);\quad \begin{cases} r = \left.\overline{\dfrac{a_n}{a_1}}\right|^{\frac{1}{n-1}}. \\ s = \dfrac{a_n^{\overline{\frac{n}{n-1}}} - a_1^{\overline{\frac{n}{n-1}}}}{a_n^{\overline{\frac{1}{n-1}}} - a_1^{\overline{\frac{1}{n-1}}}}. \end{cases}$$

$$(a_n,\ r,\ n,);\quad \begin{cases} a_1 = \dfrac{a_n}{r^{n-1}}. \\ s = \dfrac{a_n \cdot (r^n - 1)}{(r-1) r^{n-1}}. \end{cases}$$

$$(r,\ n,\ s,);\quad \begin{cases} a_1 = \dfrac{(r-1)s}{r^n - 1}. \\ a_n = \dfrac{(r-1)\, s r^{n-1}}{r^n - 1}. \end{cases}$$

$$(a_1,\ n,\ s);\quad \begin{cases} r^n - \dfrac{s}{a_1} r + \dfrac{s - a_1}{a_1} = 0. \\ a_n(s - a_n)^{n-1} - a_1(s - a_1)^{n-1} = 0. \end{cases}$$

$$(a_n,\ n,\ s);\quad \begin{cases} r^n - \dfrac{s}{s - a_n} r^{n-1} + \dfrac{a_n}{s - a_n} = 0. \\ a_1(s - a_1)^{n-1} - a_n(s - a_n)^{n-1} = 0. \end{cases}$$

$$(a_1,\ a_n,\ r);\quad \begin{cases} s = \dfrac{r\,a_n - a_1}{r-1}. \\[2ex] n = \dfrac{\log a_n - \log a_1}{\log r} + 1.\ ^{\alpha} \end{cases}$$

$$(a_1,\ r,\ s);\quad \begin{cases} a_n = \dfrac{a_1 + (r-1)s}{r}. \\[2ex] n = \dfrac{\log\,(a_1 + \overline{r-1}\,.\,s) - \log a_1}{\log r}. \end{cases}$$

$$(a_1,\ a_n,\ s);\quad \begin{cases} r = \dfrac{s - a_1}{s - a_n}. \\[2ex] n = \dfrac{\log a_n - \log a_1}{\log\,(s - a_1) - \log\,(s - a_n)} + 1. \end{cases}$$

$$(a_n,\ r,\ s);\quad \begin{cases} a_1 = r\,a_n - (r-1)s. \\[2ex] n = \dfrac{\log a_n - \log\,(r\,a_n - \overline{r-1}\,.\,s)}{\log r} + 1. \end{cases}$$

(*G.* p. 281.)

PERMUTATIONS AND COMBINATIONS.

(**24.**) The number of permutations of n things taken r at a time

$$= n(n-1)(n-2)\ldots\ldots(n-r+1).\ ^{\beta}$$

The number of combinations $= \dfrac{n\,(n-1)\,(n-2)\ldots(n-r+1)}{1\,.\ 2\ \ .\ 3\ \ \ldots\ldots\ r}$.

(*W.* 226—31; *E.* 352—60; *Bour.* 146—50.)

[α] The equations containing logarithms, are placed here on account of their connexion with the rest, although the properties of logarithms have not yet been stated.

[β] The number of permutations $= \underline{|n}_{\,r}$.

The number of combinations $= \dfrac{\underline{|n}_{\,r}}{\underline{|r}}$.

BINOMIAL THEOREM.

(25.) $(a+b)^n = a^n + \frac{n}{1} a^{n-1} b + \frac{n(n-1)}{1\,.\,2} a^{n-2} b^2 + \&c.$ [α]

the $m+1^{\text{th}}$ term being

$$\frac{n(n-1)\,(n-2)\ldots.(n-m+1)}{1\,.\,2\;.\;3\;\ldots\ldots\;m} a^{n-m} b^m.$$

If n is a positive integer, then

the number of terms $n+1$;

the coefficients of terms equally distant from both extremities of the series are equal;

the coefficient of every term is an integer;

$$1 + \frac{n}{1} + \frac{n(n-1)}{1\,.\,2} + \&c. = 2^n;$$

$$1 - \frac{n}{1} + \frac{n(n-1)}{1\,.\,2} - \&c. = 0, \text{ or}$$

$$1 + \frac{n(n-1)}{1\,.\,2} + \frac{n(n-1)\,(n-2)\,(n-3)}{1\,.\,2\;.\;3\;.\;4} + \&c.$$

$$= \frac{n}{1} + \frac{n(n-1)\,(n-2)}{1\,.\,2\;.\;3} + \&c.$$

also the theorem may be put under the following forms:

$$\frac{(a+b)^n}{1\,.\,2\,.\,3\ldots n} = \frac{a^n}{1\,.\,2\,.\,3\ldots n} + \frac{a^{n-1}\,b}{1\,.\,2\,.\,3\ldots(n-1)\,.\,1}$$

[α] $(a+b)^n = \overset{\infty}{\text{S}}_m \frac{\underline{|n}_{\,m-1}}{\underline{|m-1}} a^{n-m+1} b^{m-1}.$

The product of n simple factors, on which the binomial theorem depends, may be thus expressed:

$$\overset{n}{\text{P}}_r(x+a_r) = \overset{n+1}{\text{S}}_m x^{n-m+1} \overset{m-1,\,n}{\text{C}}_r(a_r).$$

$$+\frac{a^{n-2}b^2}{1.2.3\ldots(n-2).1.2}+\text{\&c.}$$ [β]

$$(a+b)^{2n}=(a^{2n}+b^{2n})+\frac{2n}{1}ab(a^{2n-2}+b^{2n-2})$$

$$+\frac{2n(2n-1)}{1\ .\ 2}a^2b^2(a^{2n-4}+b^{2n-4})+\text{\&c.}$$

$$+\frac{2n(2n-1)\ldots(n+2)}{1\ .\ 2\ \ldots\ n}a^{n-1}b^{n-1}(a^2+b^2)$$

$$+\frac{2n(2n-1)\ldots(n+1)}{1\ .\ 2\ \ldots\ n}a^nb^n.$$ [γ]

$$(a+b)^{2n-1}=(a^{2n-1}+b^{2n-1})+\frac{2n-1}{1}ab(a^{2n-3}+b^{2n-3})$$

$$+\frac{(2n-1)(2n-2)}{1\ .\ 2}a^2b^2(a^{2n-5}+b^{2n-5})+\text{\&c.}$$

$$+\frac{(2n-1)(2n-2)\ldots(n+1)}{1\ .\ 2\ \ldots(n-1)}a^{n-1}b^{n-1}(a+b).$$ [δ]

[β] $$\frac{(a+b)^n}{\underline{|n}}=\overset{n+1}{\mathrm{S}}_m\frac{a^{n-m+1}b^{m-1}}{\underline{|n-m+1}\,.\,\underline{|m-1}}.$$

[γ] $$(a+b)^{2n}=\overset{n}{\mathrm{S}}_m\frac{\underset{m-1}{\underline{|2n}}}{\underline{|m-1}}.(ab)^{m-1}.(a^{2n-2m+2}+b^{2n-2m+2})+\frac{\underset{n}{\underline{|2n}}}{\underline{|n}}(ab)^n.$$

[δ] $$(a+b)^{2n-1}=\overset{n}{\mathrm{S}}_m\frac{\underset{m-1}{\underline{|2n-1}}}{\underline{|m-1}}.(ab)^{m-1}.(a^{2n-2m+1}+b^{2n-2m+1}).$$

$$(a+b)^{-n}=a^{-n}-\frac{n}{1}a^{-n-1}b+\frac{n(n+1)}{1\,.\,2}a^{-n-2}b^2$$
$$-\frac{n(n+1)\,(n+2)}{1\,.\,2\,.\,3}a^{-n-3}b^3+\text{\&c.}$$ [α]

$$(a+b)^{\frac{m}{n}}=a^{\frac{m}{n}}+\frac{m}{1}a^{\frac{m}{n}-1}\,.\,\frac{b}{n}+\frac{m\,(m-n)}{1\,.\,2}a^{\frac{m}{n}-2}\,.\,\overline{\frac{b}{n}}\Big|^{2}$$
$$+\frac{m\,(m-n)\,(m-2n)}{1\,.\,2\,.\,3}a^{\frac{m}{n}-3}\,.\,\overline{\frac{b}{n}}\Big|^{3}+\text{\&c.}$$ [β]

$$(a+b)^{\frac{1}{n}}=a^{\frac{1}{n}}+\frac{1}{1}a^{\frac{1}{n}-1}\,.\,\frac{b}{n}-\frac{n-1}{1\,.\,2}a^{\frac{1}{n}-2}\,.\,\overline{\frac{b}{n}}\Big|^{2}$$
$$+\frac{(n-1)\,(2n-1)}{1\,.\,2\,.\,3}a^{\frac{1}{n}-3}\,.\,\overline{\frac{b}{n}}\Big|^{3}-\text{\&c.}$$ [γ]

$$(a+b)^{-\frac{1}{n}}=a^{-\frac{1}{n}}-\frac{1}{1}a^{-\frac{1}{n}-1}\,.\,\frac{b}{n}+\frac{1\,(1+n)}{1\,.\,2}a^{-\frac{1}{n}-2}\,.\,\overline{\frac{b}{n}}\Big|^{2}$$
$$-\frac{1\,(1+n)(1+2n)}{1\,.\,2\,.\,3}a^{-\frac{1}{n}-3}\,.\,\overline{\frac{b}{n}}\Big|^{3}+\text{\&c.}$$ [δ]

(*W*. 232—8; *G*. Chap. xii; ***Bour***. 182—90.)

[α] $(a+b)^{-n}=\overset{\infty}{\text{S}}_m(-1)^{m-1}\,.\,\frac{\underline{|n}_{\,m-1,\,1}}{\underline{|m-1}}\,.\,a^{-n-m+1}b^{m-1}.$

[β] $(a+b)^{\frac{m}{n}}=\overset{\infty}{\text{S}}_r\,\frac{\underline{|m}_{\,r-1,\,-n}}{\underline{|r-1}}\,.\,a^{\frac{m}{n}-r+1}\,.\,\overline{\frac{b}{n}}\Big|^{r-1}.$

[γ] $(a+b)^{\frac{1}{n}}=a^{\frac{1}{n}}+\overset{\infty}{\text{S}}_m(-1)^{m-1}\,.\,\frac{\underline{|n-1}_{\,m-1,\,n}}{\underline{|m}}\,.\,a^{\frac{1}{n}-m}\,.\,\overline{\frac{b}{n}}\Big|^{m}.$

[δ] $(a+b)^{-\frac{1}{n}}=a^{-\frac{1}{n}}+\overset{\infty}{\text{S}}_m(-1)^{m}\,.\,\frac{\underline{|1}_{\,m,\,n}}{\underline{|m}}\,a^{-\frac{1}{n}-m}\,.\,\overline{\frac{b}{n}}\Big|^{m}.$

(26.) $\frac{(a_1 + a_2 + a_3 + \dots + a_m)^n}{1.2.3\dots n}$ = the sum of all possible values of

$$\frac{a_1^{r_1} \;.\; a_2^{r_2} \;.\; a_3^{r_3} \;\dots\dots\; a_m^{r_m}}{(1.2.3\dots r_1)\,(1.2.3\dots r_2)\,(1.2.3\dots r_3)\dots\dots(1.2.3\dots r_m)},$$

subject to the condition that

$$r_1 + r_2 + r_3 + \dots + r_m = n;$$

the values of r_1, r_2, &c. being either 0, or a positive integer; and the partial divisor for a value 0 being 1.

(27.) $(a + a_1x + a_2x^2 + a_3x^3 + \&c.)^n$

$$= a^n + \frac{n}{1}a^{n-1}a_1x + \left.\begin{array}{r} \frac{n(n-1)}{1.\ 2}a^{n-2}a_1^2 \\ + \frac{n}{1}a^{n-1}a_2 \end{array}\right| x^2$$

$$+ \left.\begin{array}{r} \frac{n(n-1)(n-2)}{1\ .\ 2\ .\ 3}a^{n-3}a_1^3 \\ + \frac{n(n-1)}{1\ .\ 1}a^{n-2}a_1a_2 \\ + \frac{n}{1}a^{n-1}a_3 \end{array}\right| x^3 + \left.\begin{array}{r} \frac{n(n-1)(n-2)(n-3)}{1.\ 2\ .\ 3\ .\ 4}a^{n-4}a_1^4 \\ + \frac{n(n-1)(n-2)}{1.\ 2\ .\ 1}a^{n-3}a_1^2a_2 \\ + \frac{n(n-1)}{1\ .\ 1}a^{n-2}a_1a_3 \\ + \frac{n(n-1)}{1\ .\ 2}a^{n-2}a_2^2 \\ + \frac{n}{1}a^{n-1}a^4 \end{array}\right| x^4 + \&c.$$

(*Lacr. Introd. Diff. Calc.* 19—24; *Arbogast, Calc. des Deriv.* 33.)

$$\frac{x^m . a^n}{\lfloor\underline{n}} = \overset{m}{S}_r \frac{a^{n-r}}{\lfloor\underline{n-r}} . \frac{x^{m-r} . a_1^{\,r}}{\lfloor\underline{r}};$$

Continuation of Note.

$$= \overset{m}{\mathrm{S}}_r \frac{a^{n-m+r-1}}{\underline{|n-m+r-1}} \cdot \frac{\varpi^{r-1} \cdot a_1^{\;m-r+1}}{\underline{|m-r+1}}.$$

$$= \frac{a^{n-m}}{\underline{|n-m}} \cdot \frac{a_1^{\;m}}{\underline{|m}} + \frac{a^{n-m+1}}{\underline{|n-m+1}} \cdot \frac{a_1^{\;m-2}}{\underline{|m-2}} \cdot a_2$$

$$+ \frac{a^{n-m+2}}{\underline{|n-m+2}} \left\{ \frac{a_1^{\;m-4}}{\underline{|m-4}} \cdot \frac{a_2^2}{\underline{|2}} + \frac{a_1^{\;m-3}}{\underline{|m-3}} \cdot a_3 \right\}$$

$$+ \frac{a^{n-m+3}}{\underline{|n-m+3}} \left\{ \frac{a_1^{\;m-6}}{\underline{|m-6}} \cdot \frac{a_2^3}{\underline{|3}} + \frac{a_1^{\;m-5}}{\underline{|m-5}} \cdot a_2 \cdot a_3 + \frac{a_1^{\;m-4}}{\underline{|m-4}} \cdot a_4 \right\}$$

$$+ \frac{a^{n-m+4}}{\underline{|n-m+4}} \left\{ \frac{a_1^{\;m-8}}{\underline{|m-8}} \cdot \frac{a_2^4}{\underline{|4}} + \frac{a_1^{\;m-7}}{\underline{|m-7}} \cdot \frac{a_2^2}{\underline{|2}} \cdot a_3 \right.$$

$$\left. + \frac{a_1^{\;m-6}}{\underline{|m-6}} \left(\frac{a_3^2}{\underline{|2}} + a_2 \cdot a_4 \right) + \frac{a_1^{\;m-5}}{\underline{|m-5}} \cdot a_5 \right\}$$

$$+ \frac{a^{n-m+5}}{\underline{|n-m+5}} \left\{ \frac{a_1^{\;m-10}}{\underline{|m-10}} \cdot \frac{a_2^5}{\underline{|5}} + \frac{a_1^{\;m-9}}{\underline{|m-9}} \cdot \frac{a_2^3}{\underline{|3}} \cdot a_3 \right.$$

$$+ \frac{a_1^{\;m-8}}{\underline{|m-8}} \left(a_2 \cdot \frac{a_3^2}{\underline{|2}} + \frac{a_2^2}{\underline{|2}} \cdot a_4 \right)$$

$$\left. + \frac{a_1^{\;m-7}}{\underline{|m-7}} (a_3 \cdot a_4 + a_2 \cdot a_5) + \frac{a_1^{\;m-6}}{\underline{|m-6}} \cdot a_6 \right\}$$

$$+ \frac{a^{n-m+6}}{\underline{|n-m+6}} \left\{ \frac{a_1^{\;m-12}}{\underline{|m-12}} \cdot \frac{a_2^6}{\underline{|6}} + \frac{a_1^{\;m-11}}{\underline{|m-11}} \cdot \frac{a_2^4}{\underline{|4}} \cdot a_3 \right.$$

$$+ \frac{a_1^{\;m-10}}{\underline{|m-10}} \left(\frac{a_2^2}{\underline{|2}} \cdot \frac{a_3^2}{\underline{|2}} + \frac{a_2^3}{\underline{|3}} \cdot a^4 \right)$$

$$+ \frac{a_1^{\;m-9}}{\underline{|m-9}} \left(\frac{a_3^3}{\underline{|3}} + a_2 \cdot a_3 \cdot a_4 + \frac{a_2^2}{\underline{|2}} \cdot a_5 \right)$$

$$\left. + \frac{a_1^{\;m-8}}{\underline{|m-8}} \left(\frac{a_4^2}{\underline{|2}} + a_3 \cdot a_5 + a_2 \cdot a_6 \right) + \frac{a_1^{\;m-7}}{\underline{|m-7}} \cdot a_7 \right\}$$

Continuation of Note.

$$+\frac{a^{n-m+7}}{\lfloor\underline{n-m+7}}\left\{\frac{a_1^{m-14}}{\lfloor\underline{m-14}}\cdot\frac{a_2^7}{\lfloor\underline{7}}+\frac{a_1^{m-13}}{\lfloor\underline{m-13}}\cdot\frac{a_2^5}{\lfloor\underline{5}}\cdot a_3\right.$$

$$+\frac{a_1^{m-12}}{\lfloor\underline{m-12}}\left(\frac{a_2^3}{\lfloor\underline{3}}\cdot\frac{a_3^2}{\lfloor\underline{2}}+\frac{a_2^4}{\lfloor\underline{4}}\cdot a_4\right)$$

$$+\frac{a_1^{m-11}}{\lfloor\underline{m-11}}\left(a_2\cdot\frac{a_3^3}{\lfloor\underline{3}}+\frac{a_2^2}{\lfloor\underline{2}}\cdot a_3\cdot a_4+\frac{a_2^3}{\lfloor\underline{3}}\cdot a_5\right)$$

$$+\frac{a_1^{m-10}}{\lfloor\underline{m-10}}\left(\frac{a_3^2}{\lfloor\underline{2}}\cdot a_4+a_2\left[\frac{a_4^2}{\lfloor\underline{2}}+a_3\cdot a_5\right]+\frac{a_2^2}{\lfloor\underline{2}}\cdot a_6\right)$$

$$\left.+\frac{a_1^{m-9}}{\lfloor\underline{m-9}}(a_4\cdot a_5+a_3\cdot a_6+a_2\cdot a_7)+\frac{a_1^{m-8}}{\lfloor\underline{m-8}}\cdot a_8\right\}$$

$$+\frac{a^{n-m+8}}{\lfloor\underline{n-m+8}}\left\{\frac{a_1^{m-16}}{\lfloor\underline{m-16}}\cdot\frac{a_2^8}{\lfloor\underline{8}}+\frac{a_1^{m-15}}{\lfloor\underline{m-15}}\cdot\frac{a_2^6}{\lfloor\underline{6}}\cdot a_3\right.$$

$$+\frac{a_1^{m-14}}{\lfloor\underline{m-14}}\left(\frac{a_2^4}{\lfloor\underline{4}}\cdot\frac{a_3^2}{\lfloor\underline{2}}+\frac{a_2^5}{\lfloor\underline{5}}\cdot a_4\right)$$

$$+\frac{a_1^{m-13}}{\lfloor\underline{m-13}}\left(\frac{a_2^2}{\lfloor\underline{2}}\cdot\frac{a_3^3}{\lfloor\underline{3}}+\frac{a_2^3}{\lfloor\underline{3}}\cdot a_3\cdot a_4+\frac{a_2^4}{\lfloor\underline{4}}\cdot a_5\right)$$

$$+\frac{a_1^{m-12}}{\lfloor\underline{m-12}}\left(\frac{a_3^4}{\lfloor\underline{4}}+a_2\cdot\frac{a_3^2}{\lfloor\underline{2}}\cdot a_4+\frac{a_2^2}{\lfloor\underline{2}}\left[\frac{a_4^2}{\lfloor\underline{2}}+a_3\cdot a_5\right]+\frac{a_2^3}{\lfloor\underline{3}}\cdot a_6\right)$$

$$+\frac{a_1^{m-11}}{\lfloor\underline{m-11}}\left(a_3\cdot\frac{a_4^2}{\lfloor\underline{2}}+\frac{a_3^2}{\lfloor\underline{2}}\cdot a_5+a_2[a_4\cdot a_5+a_3\cdot a_6]+\frac{a_2^2}{\lfloor\underline{2}}\cdot a_7\right)$$

$$+\frac{a_1^{m-10}}{\lfloor\underline{m-10}}\left(\frac{a_5^2}{\lfloor\underline{2}}+a_4\cdot a_6+a_3\cdot a_7+a_2\cdot a_8\right)$$

$$\left.+\frac{a_1^{m-9}}{\lfloor\underline{m-9}}\cdot a_9\right\}+\ \&c$$

Continuation of Note.

$$\frac{\varpi \cdot a^n}{\underline{|n}} = \frac{a^{n-1}}{\underline{|n-1}} \cdot a_1 .$$

$$\frac{\varpi^2 \cdot a^n}{\underline{|n}} = \frac{a^{n-2}}{\underline{|n-2}} \cdot \frac{a_1^2}{\underline{|2}} + \frac{a^{n-1}}{\underline{|n-1}} \cdot a_2 .$$

$$\frac{\varpi^3 \cdot a^n}{\underline{|n}} = \frac{a^{n-3}}{\underline{|n-3}} \cdot \frac{a_1^3}{\underline{|3}} + \frac{a^{n-2}}{\underline{|n-2}} \cdot a_1 \cdot a_2 + \frac{a^{n-1}}{\underline{|n-1}} \cdot a_3 .$$

$$\begin{aligned} \frac{\varpi^4 \cdot a^n}{\underline{|n}} = {} & \frac{a^{n-4}}{\underline{|n-4}} \cdot \frac{a_1^4}{\underline{|4}} + \frac{a^{n-3}}{\underline{|n-3}} \cdot \frac{a_1^2}{\underline{|2}} \cdot a_2 \\ & + \frac{a^{n-2}}{\underline{|n-2}} \left\{ \frac{a_2^2}{\underline{|2}} + a_1 \cdot a_3 \right\} + \frac{a^{n-1}}{\underline{|n-1}} \cdot a_4 . \end{aligned}$$

$$\begin{aligned} \frac{\varpi^5 \cdot a^n}{\underline{|n}} = {} & \frac{a^{n-5}}{\underline{|n-5}} \cdot \frac{a_1^5}{\underline{|5}} + \frac{a^{n-4}}{\underline{|n-4}} \cdot \frac{a_1^3}{\underline{|3}} \cdot a_2 \\ & + \frac{a^{n-3}}{\underline{|n-3}} \left\{ a_1 \cdot \frac{a_2^2}{\underline{|2}} + \frac{a_1^2}{\underline{|2}} \cdot a_3 \right\} \\ & + \frac{a^{n-2}}{\underline{|n-2}} \left\{ a_2 \cdot a_3 + a_1 \cdot a_4 \right\} + \frac{a^{n-1}}{\underline{|n-1}} \cdot a_5 . \end{aligned}$$

$$\begin{aligned} \frac{\varpi^6 \cdot a^n}{\underline{|n}} = {} & \frac{a^{n-6}}{\underline{|n-6}} \cdot \frac{a_1^6}{\underline{|6}} + \frac{a^{n-5}}{\underline{|n-5}} \cdot \frac{a_1^4}{\underline{|4}} \cdot a_2 \\ & + \frac{a^{n-4}}{\underline{|n-4}} \left\{ \frac{a_1^2}{\underline{|2}} \cdot \frac{a_2^2}{\underline{|2}} + \frac{a_1^3}{\underline{|3}} \cdot a_3 \right\} \\ & + \frac{a^{n-3}}{\underline{|n-3}} \left\{ \frac{a_2^3}{\underline{|3}} + a_1 \cdot a_2 \cdot a_3 + \frac{a_1^2}{\underline{|2}} \cdot a_4 \right\} \\ & + \frac{a^{n-2}}{\underline{|n-2}} \left\{ \frac{a_3^2}{\underline{|2}} + a_2 \cdot a_4 + a_1 \cdot a_5 \right\} + \frac{a^{n-1}}{\underline{|n-1}} \cdot a_6 . \end{aligned}$$

$$\frac{\varpi^7 \cdot a^n}{\underline{|n}} = \frac{a^{n-7}}{\underline{|n-7}} \cdot \frac{a_1^7}{\underline{|7}} + \frac{a^{n-6}}{\underline{|n-6}} \cdot \frac{a_1^5}{\underline{|5}} \cdot a_2$$

Continuation of Note.

$$+\frac{a^{n-5}}{\underline{|n-5}}\left\{\frac{a_1^3}{\underline{|3}}\cdot\frac{a_2^2}{\underline{|2}}+\frac{a_1^4}{\underline{|4}}\cdot a_3\right\}$$

$$+\frac{a^{n-4}}{\underline{|n-4}}\left\{a_1\cdot\frac{a_2^3}{\underline{|3}}+\frac{a_1^2}{\underline{|2}}\cdot a_2\cdot a_3+\frac{a_1^3}{\underline{|3}}\cdot a_4\right\}$$

$$+\frac{a^{n-3}}{\underline{|n-3}}\left\{\frac{a_2^2}{\underline{|2}}\cdot a_3+a_1\left(\frac{a_3^2}{\underline{|2}}+a_2\cdot a_4\right)+\frac{a_1^2}{\underline{|2}}\cdot a_5\right\}$$

$$+\frac{a^{n-2}}{\underline{|n-2}}\left\{a_3\cdot a_4+a_2\cdot a_5+a_1\cdot a_6\right\}+\frac{a^{n-1}}{\underline{|n-1}}\cdot a_7\,.$$

$$\frac{\varpi^8\cdot a^n}{\underline{|n}}=\frac{a^{n-8}}{\underline{|n-8}}\cdot\frac{a_1^8}{\underline{|8}}+\frac{a^{n-7}}{\underline{|n-7}}\cdot\frac{a_1^6}{\underline{|6}}\cdot a_2$$

$$+\frac{a^{n-6}}{\underline{|n-6}}\left\{\frac{a_1^4}{\underline{|4}}\cdot\frac{a_2^2}{\underline{|2}}+\frac{a_1^5}{\underline{|5}}\cdot a_3\right\}$$

$$+\frac{a^{n-5}}{\underline{|n-5}}\left\{\frac{a_1^2}{\underline{|2}}\cdot\frac{a_2^3}{\underline{|3}}+\frac{a_1^3}{\underline{|3}}\cdot a_2\cdot a_3+\frac{a_1^4}{\underline{|4}}\cdot a_4\right\}$$

$$+\frac{a^{n-4}}{\underline{|n-4}}\left\{\frac{a_2^4}{\underline{|4}}+a_1\cdot\frac{a_2^2}{\underline{|2}}\cdot a_3+\frac{a_1^2}{\underline{|2}}\left(\frac{a_3^2}{\underline{|2}}+a_2\cdot a_4\right)+\frac{a_1^3}{\underline{|3}}\cdot a_5\right\}$$

$$+\frac{a^{n-3}}{\underline{|n-3}}\left\{a_2\cdot\frac{a_3^2}{\underline{|2}}+\frac{a_2^2}{\underline{|2}}\cdot a_4+a_1\left(a_3\cdot a_4+a_2\cdot a_5\right)+\frac{a_1^2}{\underline{|2}}\cdot a_6\right\}$$

$$+\frac{a^{n-2}}{\underline{|n-2}}\left\{\frac{a_4^2}{\underline{|2}}+a_3\cdot a_5+a_2\cdot a_6+a_1\cdot a_7\right\}+\frac{a^{n-1}}{\underline{|n-1}}\cdot a_8\,.$$

In this theorem, n has been considered a positive integer; if an expansion be required for any value of n, we must expand in a similar manner

$$\varpi^m\cdot a^n=\overset{m}{S}_r\,\underline{|n}_{\,r}\cdot a^{n-r}\cdot\frac{\varpi^{m-r}\cdot a_1^{\,r}}{\underline{|r}}\,.$$

INDETERMINATE COEFFICIENTS.

(28.) If $a + a_1 x + a_2 x^2 + \&c. = b + b_1 x + b_2 x^2 + \&c.$ for every value of x, then

$$a = b, \quad a_1 = b_1, \quad \&c.$$ [a]

If $a + a_1 x + a_2 x^2 + \&c. = 0$, then

$$a = 0 = a_1 = \&c.$$

(*W.* 346; *Bour.* 187, 8.)

LOGARITHMS.

(29.) Theory of logarithms, and logarithmic tables.

(*E.* 220—55; *Bour.* 209—24.)

Let $\log x$ represent the logarithm of x to any base;

$\log_a x$, the logarithm of x to the base a.

$a^{\log_a x} = x$.

$\log x + \log y = \log xy$.

---

[a] If $\overset{n}{\mathrm{S}}_m a_{m-1} x^{m-1} = \overset{n}{\mathrm{S}}_m b_{m-1} x^{m-1}$,

$$a_{m-1} = b_{m-1}.$$

This is only a particular case of the following more general theorems, which may be demonstrated in a similar manner:

If $\overset{n}{\mathrm{S}}_m a_{m-1} \overset{m-1}{\mathrm{P}}_r u_r = \overset{n}{\mathrm{S}}_m b_{m-1} \overset{m-1}{\mathrm{P}}_r u_r$,

whatever may be the form or value of u_r, then

$$a_{m-1} = b_{m-1}.$$

If $\overset{n}{\mathrm{S}}_m a_{m-1} \underline{\lfloor x}_{m-1,\,r} = \overset{n}{\mathrm{S}}_m b_{m-1} \underline{\lfloor x}_{m-1,\,r}$, then

$$a_{m-1} = b_{m-}$$

Either of these equations may be taken as the definition of a logarithm: the former is most usually adopted; and the various properties are therefore given in the order in which they most naturally follow on that supposition.

$$\mathrm{Log}_a a = 1.$$

$$\log 1 = 0.$$ [β]

$$a^x = b^{x \cdot \log_b a}.$$

$$\log_b x = \frac{\log_a x}{\log_a b}.$$

$$\log_a b \,.\, \log_b a = 1.$$

$$\log x_1 + \log x_2 + \log x_3 + \&c. = \log (x_1 \,.\, x_2 \,.\, x_3 \,.\, \&c.)$$ [γ]

$$\log x - \log y = \log \frac{x}{y}.$$

$$\log x^{\frac{m}{n}} = \frac{m}{n} \log x.$$

$$\log_a x = \frac{\dfrac{x-1}{1} - \dfrac{(x-1)^2}{2} + \dfrac{(x-1)^3}{3} - \&c.}{\dfrac{a-1}{1} - \dfrac{(a-1)^2}{2} + \dfrac{(a-1)^3}{3} - \&c.}.$$ [δ]

[β] It will be hereafter seen that unity has an infinite number of logarithms, of which the above is the only *possible* one.

[γ] $\overset{n}{\mathrm{S}}_m \log x_m = \log \overset{n}{\mathrm{P}}_m x_m.$

[δ] $$\log_a x = \frac{\overset{\infty}{\mathrm{S}}_m (-1)^{m-1} \dfrac{(x-1)^m}{m}}{\overset{\infty}{\mathrm{S}}_m (-1)^{m-1} \dfrac{(a-1)^m}{m}}.$$

$$\log_\epsilon x = \frac{x-1}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \&c. \quad \alpha$$

where ϵ is such a number that

$$\frac{\epsilon-1}{1} - \frac{(\epsilon-1)^2}{2} + \frac{(\epsilon-1)^3}{3} - \&c. = 1. \quad \beta$$

$$\log_a x = \frac{1}{\log_\epsilon a}\left\{\frac{x-1}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \&c.\right\}. \quad \gamma$$

$$a^x = 1 + \frac{\log_\epsilon a \,.\, x}{1} + \frac{(\log_\epsilon a \,.\, x)^2}{1\,.\,2} + \frac{(\log_\epsilon a \,.\, x)^3}{1\,.\,2\,.\,3} + \&c. \quad \delta$$

$$\epsilon^x = 1 + \frac{x}{1} + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \&c. \quad \varepsilon$$

$$\epsilon = 1 + \frac{1}{1} + \frac{1}{1\,.\,2} + \frac{1}{1\,.\,2\,.\,3} + \&c. \quad \theta$$

$$= 2,71828182845904523536028 \;\&c.$$

The number ϵ is incommensurable. (*Bour. Appendix.*)

α $\log_\epsilon x = \overset{\infty}{S}_m (-1)^{m-1}\dfrac{(x-1)^m}{m}.$

β $\overset{\infty}{S}_m (-1)^{m-1}\dfrac{(\epsilon-1)^m}{m} = 1.$

γ $\log_a x = \dfrac{1}{\log_\epsilon a}\overset{\infty}{S}_m (-1)^{m-1}\dfrac{(x-1)^m}{m}.$

δ $a^x = \overset{\infty}{S}_m \dfrac{(\log_\epsilon a \,.\, x)^{m-1}}{\underline{|m-1}}.$

ε $\epsilon^x = \overset{\infty}{S}_m \dfrac{x^{m-1}}{\underline{|m-1}}.$

θ $\epsilon = \overset{\infty}{S}_m \dfrac{1}{\underline{|m-1}}.$

$$\log_{10} x = \frac{\log_\epsilon x}{\log_\epsilon 10} = \log_{10} \epsilon . \log_\epsilon x ;$$

$$= 0,4342944819032518276511289 . \log_\epsilon x .$$

$$\log_\epsilon 10 = 2,3025850929940456840179914 \ \&c.$$

$$\log_\epsilon (1 + x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \&c. \quad {}^{\alpha}$$

$$\log_\epsilon (1 - x) = -\left(\frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \&c.\right). \quad {}^{\beta}$$

$$\log_\epsilon \frac{1 + x}{1 - x} = 2\left\{\frac{x}{1} + \frac{x^3}{3} + \frac{x^5}{5} + \&c.\right\}. \quad {}^{\gamma}$$

$$\log_\epsilon \frac{x + 1}{x - 1} = 2\left\{\frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \&c.\right\}. \quad {}^{\delta}$$

$$\log_\epsilon x = \log_\epsilon a + 2\left\{\frac{x - a}{x + a} + \frac{1}{3}\left(\frac{x - a}{x + a}\right)^3 + \frac{1}{5}\left(\frac{x - a}{x + a}\right)^5 + \&c\right\}. \quad {}^{\epsilon}$$

[α] $\log_\epsilon (1 + x) = \overset{\infty}{\mathrm{S}}_m (-1)^{m-1} \frac{x^m}{m}$.

[β] $\log_\epsilon (1 - x) = -\overset{\infty}{\mathrm{S}}_m \frac{x^m}{m}$.

[γ] $\log_\epsilon \frac{1 + x}{1 - x} = 2\overset{\infty}{\mathrm{S}}_m \frac{x^{2m-1}}{2m-1}$.

[δ] $\log_\epsilon \frac{x + 1}{x - 1} = 2\overset{\infty}{\mathrm{S}}_m \frac{1}{(2m-1)x^{2m-1}}$.

[ε] $\log_\epsilon x = \log_\epsilon a + 2\overset{\infty}{\mathrm{S}}_m \frac{1}{2m-1}\left(\frac{x - a}{x + a}\right)^{2m-1}$.

$$\log_e x = 2\left\{\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \&c.\right\}. \quad \alpha$$

$$\log_e x = \log_e(x-1) + 2\left\{\frac{1}{2x-1} + \frac{1}{3(2x-1)^3} + \&c.\right\}. \quad \beta$$

$$\log_e x = n\left\{\frac{x^{\frac{1}{n}}-1}{1} - \frac{(x^{\frac{1}{n}}-1)^2}{2} + \frac{(x^{\frac{1}{n}}-1)^3}{3} - \&c.\right\}. \quad \gamma$$

$$\log_e x = n\left\{\frac{(1-x^{-\frac{1}{n}})}{1} + \frac{(1-x^{-\frac{1}{n}})^2}{2} + \frac{(1-x^{-\frac{1}{n}})^3}{3} + \&c.\right\}. \quad \delta$$

(*Lagrange, Calc. des Fonctions, Leçon* 4$^{\text{me}}$.)

$$\log_e x = \frac{1}{n}\left\{\frac{x^n-1}{x^n+1} + \frac{x^{2n}-1}{2(x^n+1)^2} + \frac{x^{3n}-1}{3(x^n+1)^3} + \&c.\right\}. \quad \varepsilon$$

(*Bouvier, Ann. de Math. Tome* 16.)

$$\alpha \quad \log_e x = 2\,\overset{\infty}{\mathrm{S}}_m \frac{1}{2m-1}\cdot\left(\frac{x-1}{x+1}\right)^{2m-1}.$$

$$\beta \quad \log_e x = \log_e(x-1) + 2\,\overset{\infty}{\mathrm{S}}_m \frac{1}{(2m-1)\,.\,(2x-1)^{2m-1}}.$$

$$\gamma \quad \log_e x = n\,.\,\overset{\infty}{\mathrm{S}}_m (-1)^{m-1}\frac{(x^{\frac{1}{n}}-1)^m}{m}.$$

$$\delta \quad \log_e x = n\,.\,\overset{\infty}{\mathrm{S}}_m \frac{(1-x^{-\frac{1}{n}})^m}{m}.$$

$$\varepsilon \quad \log_e x = \frac{1}{n}\,\overset{\infty}{\mathrm{S}}_m \frac{x^{mn}-1}{m(x^n+1)^m}.$$

$$\log_\varepsilon x = \frac{x - x^{-1}}{1} - \frac{x^2 - x^{-2}}{2} + \frac{x^3 - x^{-3}}{3} - \&c.\Big\}$$ [α]

$$\log_\varepsilon (1+x) = (1+x)^{-n}\Big\{x - \left(\frac{1}{2} - \frac{n}{1}\cdot\frac{1}{1}\right)x^2$$

$$+ \left(\frac{1}{3} - \frac{n}{1}\cdot\frac{1}{2} + \frac{n(n-1)}{1\,.\,2}\cdot\frac{1}{1}\right)x^3 - \left(\frac{1}{4} - \frac{n}{1}\cdot\frac{1}{3}\right.$$

$$\left. + \frac{n(n-1)}{1\,.\,2}\cdot\frac{1}{2} - \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\cdot\frac{1}{1}\right)x^4 + \&c.\Big\}.$$ [β]

$$\log_\varepsilon (x+a+b) = \log_\varepsilon (a - \overline{a+b}) + \log_\varepsilon (x+a) + \log_\varepsilon (x+b)$$

$$- \log_\varepsilon (x-a) - \log_\varepsilon (x-b)$$

$$+ 2\left\{\frac{ab(a+b)}{x^3 - x(a^2+ab+b^2)} + \frac{1}{3}\left(\frac{ab(a+b)}{x^3 - x(a^2+ab+b^2)}\right)^3 + \&c.\right\}.$$ [γ]

(*Delambre, Introd. aux Tables de Borda.*)

$$\log_\varepsilon x = 2\log_\varepsilon (x-1) - \log_\varepsilon (x-2)$$

$$- 2\left\{\frac{1}{2x^2 - 4x - 1} + \frac{1}{3(2x^2-4x-1)^3} + \&c.\right\}.$$ [δ]

[α] $\log_\varepsilon x = \overset{\infty}{\mathrm{S}}_m (-1)^{m-1}\dfrac{x^m - x^{-m}}{m}.$

[β] $\log_\varepsilon (1+x) = (1+x)^{-n}.\overset{\infty}{\mathrm{S}}_m x^m \overset{m}{\mathrm{S}}_r (-1)^{m-r}\dfrac{\lfloor\!\underline{n}_{\,r-1}}{\lfloor\!\underline{r-1}\,.(m-r+1)}.$

[γ] $+2\overset{\infty}{\mathrm{S}}_m \dfrac{1}{2m-1}\left(\dfrac{ab(a+b)}{x^3 - x(a^2+ab+b^2)}\right)^{2m-1}.$

[δ] $-2\overset{\infty}{\mathrm{S}}_m \dfrac{1}{(2m-1)(2x^2-4x-1)^{2m-1}}.$

$$\log_e(x+2)-2\log_e(x+1)+2\log_e(x-1)-\log_e(x-2)$$
$$=2\left\{\frac{2}{x^3-3x}+\frac{1}{3}\left(\frac{2}{x^3-3x}\right)^3+\&c.\right\}.\ ^{\alpha}$$

$$\log_e(x+5)-\log_e(x+4)-\log_e(x+3)+2\log_e x$$
$$-\log_e(x-3)-\log_e(x-4)+\log_e(x-5)$$
$$=2\left\{\frac{72}{x^4-25x^2+72}+\frac{1}{3}\left(\frac{72}{x^4-25x^2+72}\right)^3+\&c.\right\}$$

$$-\log_e(x+6)+2\log_e(x+5)-\log_e(x+3)-\log_e(x+2)$$
$$+2\log_e x-\log_e(x-1)$$
$$=2\left\{\frac{18}{x^4+10x^3+25x^2-18}+\frac{1}{3}\left(\frac{18}{x^4+10x^3+25x^2-18}\right)^3+\&c.\right\}.$$

(*Lavernède, Ann. de Math.*)

To find the logarithm of any number from the tabulated logarithm of that number without the last digit;

$$\log_{10}(10x+a)=1+\log x$$
$$+(\log_{10}e)\left\{\frac{a}{10x}-\frac{1}{2}\left(\frac{a}{10x}\right)^2+\frac{1}{3}\left(\frac{a}{10x}\right)^3+\&c.\right\}.$$

(*Cagnoli, Trig.*)

CONTINUED FRACTIONS.

(30.) The general form of a continued fraction is

$$\frac{a}{b}=\cfrac{1}{c_1+\cfrac{1}{c_2+\cfrac{1}{c_3+\&c.}}}$$

$$^{\alpha}\quad =2\overset{\infty}{S}_m\frac{1}{2m-1}\left(\frac{2}{x^3-3x}\right)^{2m-1}.$$

which may for convenience be thus expressed:

$$\frac{a}{b} = \frac{1}{c_1 +}\frac{1}{c_2 +}\frac{1}{c_3 + \ldots +}\frac{1}{c_n}.$$

$$= \underset{1}{\{} \underset{2}{\{} \ldots \underset{n}{\{} c_n \underset{n}{\}}^{-1} + c_{n-1} \underset{n-1}{\}}^{-1} + \ldots + c_2 \underset{2}{\}}^{-1} + c_1 \underset{1}{\}}^{-1},$$

by inverting the order of the terms.

n is a finite quantity only when $\frac{a}{b}$ is rational.

Let the m^{th} converging fraction be $\frac{A_m}{B_m}$.

$$\frac{A_1}{B_1} = \frac{1}{c_1},$$

$$\frac{A_2}{B_2} = \frac{1}{c_1 +}\frac{1}{c_2} = \frac{c_2}{c_1 \cdot c_2 + 1},$$

$$\frac{A_3}{B_3} = \frac{1}{c_1 +}\frac{1}{c_2 +}\frac{1}{c_3} = \frac{c_2 \cdot c_3 + 1}{c_1 \cdot c_2 \cdot c_3 + c_3 + c_1},$$

$$\&c. = \quad \&c. \quad = \quad \&c.$$

$$\frac{A_{m+2}}{B_{m+2}} = \frac{A_{m+1} \cdot c_{m+2} + A_m}{B_{m+1} \cdot c_{m+2} + B_m}, \qquad (1).$$

$$\frac{a}{b} < \frac{A_{2m-1}}{B_{2m-1}}, \text{ and } > \frac{A_{2m}}{B_{2m}}.$$

$$A_{2m-1} \cdot B_{2m} - A_{2m} \cdot B_{2m-1} = 1,$$

$$A_{2m} \cdot B_{2m+1} - A_{2m+1} \cdot B_{2m} = -1.$$

$$\frac{a}{b} \sim \frac{A_m}{B_m} < \frac{1}{B_m \cdot B_{m+1}} < \frac{1}{B_m^2};$$

$$> \frac{1}{B_m (B_m + B_{m+1})}.$$

$$\frac{a}{b} \sim \frac{A_m}{B_m} > \frac{a}{b} \sim \frac{A_{m+1}}{B_{m+1}}.$$

If p, q are any numbers whatever $< A_m$, B_m, respectively,

$$\frac{p}{q} \sim \frac{a}{b} > \frac{A_m}{B_m} \sim \frac{a}{b}.$$

If $A_{2m+2}B_{2m} - A_{2m}B_{2m+2} = n$, then

$$\frac{A_{2m}}{B_{2m}}, \quad \frac{A_{2m}+A_{2m+1}}{B_{2m}+B_{2m+1}}, \quad \frac{A_{2m}+2A_{2m+1}}{B_{2m}+2B_{2m+1}}, \text{ \&c.}$$

$$\frac{A_{2m}+(n-1)B_{2m+1}}{B_{2m}+(n-1)B_{2m+1}}, \quad \frac{A_{2m+2}}{B_{2m+2}},$$

form an increasing series, each term of which is $< \frac{a}{b}$.

If $A_{2m-1}B_{2m+1} - A_{2m+1}B_{2m-1} = n$, then

$$\frac{A_{2m-1}}{B_{2m-1}}, \quad \frac{A_{2m-1}+A_{2m}}{B_{2m-1}+B_{2m}}, \quad \frac{A_{2m-1}+2A_{2m}}{B_{2m-1}+2B_{2m}}, \text{ \&c.}$$

$$\frac{A_{2m-1}+(n-1)A_{2m}}{B_{2m-1}+(n-1)B_{2m}}, \quad \frac{A_{2m+1}}{B_{2m+1}},$$

form a decreasing series, each term of which is $> \frac{a}{b}$.

No rational fraction whose denominator lies between the denominators of any two adjacent terms of these series can be inserted between those terms.

$$\frac{A_m}{B_m} = \frac{1}{B_1} - \frac{1}{B_1 \cdot B_2} + \frac{1}{B_2 \cdot B_3} + \dots + \frac{(-1)^m}{B_m \cdot B_{m+1}}.$$

To approximate to the value of a fraction, whose numerator and denominator are high numbers: obtain c_1, c_2, c_3, &c. by actual division, and the series of converging fractions may be obtained from equation (1).

A continued fraction may frequently be simplified by the introduction of negative quotients; for

$$a+\cfrac{1}{1+\cfrac{1}{b}}=a+1-\frac{1}{b-1}.$$

(*G.* Chap. xxxi; *G. A.* 12—20; *E. Add.* Art 1; *Lagrange, Equ. Num.* Chap. vi, Art. 1, 3.)

To reduce $\frac{b+b_1+b_2+\&c.}{a+a_1+a_2+\&c.}$ to a continued fraction: dividing the denominator by the numerator and taking one term of the quotient, we have

$$\frac{a+a_1+\&c.}{b+b_1+\&c.}=\frac{a}{b}+\frac{\left(a_1-\frac{a}{b}.b_1\right)+\left(a_2-\frac{a}{b}.b_2\right)+\&c.}{b\quad+\quad b_1\quad+\&c.}:$$

Assume $a_1-\frac{a}{b}.b_1=c$, $a_2-\frac{a}{b}.b_2=c_1$, &c. = &c.,

proceeding as before, the second quotient will be

$$\frac{b+b_1+\&c.}{c+c_1+\&c.}=\frac{b}{c}+\frac{\left(b_1-\frac{b}{c}.c_1\right)+\left(b_2-\frac{b}{c}.c_2\right)+\&c.}{c\quad+\quad c_1\quad+\&c.}.$$

In the same manner, assume

$$b_1+\frac{b}{c}.c_1=d,\ b_2+\frac{b}{c}c_2=d_1,\ \&c.=\&c.$$

and proceed as before: we at length obtain

$$\frac{b+b_1+b_2+\&c.}{a+a_1+a_2+\&c.}=\cfrac{1}{\frac{a}{b}+\cfrac{1}{\frac{b}{c}+\cfrac{1}{\frac{c}{d}+\&c.}}}\qquad(G.\ A.\ 11.)$$

Periodic continued fractions. The value of every periodic fraction may be determined by the solution of a quadratic equation; and the root of every quadratic equation may be expressed by a periodic fraction.

If $\frac{a}{b} = \frac{\dot{1}}{\dot{c}}$,[α]

$$\left.\overline{\frac{a}{b}}\right]^2 + c . \frac{a}{b} - 1 = 0, \text{ and}$$

$$\frac{a}{b} = \tfrac{1}{2}\left(-c \pm \left.\overline{c^2 + 4}\right|^{\frac{1}{2}}\right).$$

If $a = \frac{\dot{1}}{c_1 +} \frac{\dot{1}}{c_2}$,

$$\left.\overline{\frac{a}{b}}\right]^2 + c_2 . \frac{a}{b} - \frac{c_2}{c_1} = 0.$$

If $\frac{a}{b} = \frac{\dot{1}}{c_1 +} \frac{1}{c_2 +} \frac{\dot{1}}{c_3}$,

$$\left.\overline{\frac{a}{b}}\right]^2 + \frac{c_1 . c_2 . c_3 + c_1 - c_2 + c_3}{c_1 c_2 + 1} . \frac{a}{b} - \frac{c_2 . c_3 + 1}{c_1 . c_2 + 1} = 0.$$

(*G. A.* 21, 2; *B.* 147—9.)

To reduce the roots of a quadratic equation to continued fractions: let the equation be

$$ax^2 + bx + c = 0, \quad (1)$$

in which $b^2 - 4ac > 0$.

[α] The *full point* is placed over the first and last fraction of the period, as in circulating decimals, to denote the extent of the period.

from (1), $$x = \frac{1}{2a}(-b + \overline{b^2 - 4ac}\,|^{\frac{1}{2}});$$

assume $x = e + \dfrac{1}{x_1}$, x being $> e$ but $< e + 1$, then

$$a_1 x_1^2 + b_1 x + a = 0, \qquad (2)$$

in which equation $a_1 = ae^2 + be + c$,

$$b_1 = 2ae + b;$$

from (2), $$x_1 = \frac{1}{2a_1}(-b_1 + \overline{b_1^2 - 4a\,.\,a_1}\,|^{\frac{1}{2}}),$$

$$= e_1 + \frac{1}{x_2}, \text{ and so on.}$$

$$b^2 - 4ac = b_1^2 - 4a\,.\,a_1 = b_2^2 - 4a_1\,.\,a_2 = \text{\&c.}$$
$$= d, \text{ suppose; then}$$

$$\begin{array}{l|l|l}
b_1 = 2a\,.\,e + b & 2a_1 = \dfrac{b_1^2 - d}{2a} & \dfrac{\sqrt{d} - b_1}{2a_1} > e_1 \text{ and } < e_1 + 1; \\
b_2 = 2a_1\,.\,e_1 + b_1 & 2a_2 = \dfrac{b_2^2 - d}{2a_1} & \dfrac{\sqrt{d} - b_2}{2a_2} > e_2 \text{ and } < e_2 + 1; \\
b_3 = 2a_2\,.\,e_2 + b_2 & 2a_3 = \dfrac{b_3^2 - d}{2a_2} & \dfrac{\sqrt{d} - b_3}{2a_3} > e_3 \text{ and } < e_3 + 1. \\
\text{\&c.} & \text{\&c.} & \text{\&c.}
\end{array}$$

and $$x = e + \frac{1}{e_1 +}\,\frac{1}{e_2 +}\,\frac{1}{e_3 +} \text{ \&c.}$$

In the same manner we may find the value of the other root of (1), viz.

$$\frac{1}{2a}(-b - \overline{b^2 - 4ac}\,|^{\frac{1}{2}}).$$

The values of e_1, e_2, &c. are the same in this case, but they occur in an inverted order.

(*G. A.* 23; *Lagr. Equ. Num. Ch.* 6. *Art.* 2; *Legendre, Theor. Nomb.* 59—74.)

(31.) To find the value of $\sqrt{c}$: the equation (1) and the above formulæ become respectively

$$x^2-c=0,$$

$$\begin{array}{l|l|l}
b_1=1\,.\,e-0 & a_1=\dfrac{c-b_1^2}{1} & \dfrac{\sqrt{c}+b_1}{a_1}>e_1 \text{ and} <e_1+1; \\[2ex]
b_2=a_1\,.\,e_1-b_1 & a_2=\dfrac{c-b_2^2}{a_1} & \dfrac{\sqrt{c}+b_2}{a_2}>e_2 \text{ and} <e_2+1; \\[2ex]
b_3=a_2\,.\,e_2-b_2 & a_3=\dfrac{c-b_3^2}{a_2} & \dfrac{\sqrt{c}+b_3}{a_3}>e_3 \text{ and} <e_3+1. \\[2ex]
\&c. & \&c. & \&c.
\end{array}$$

$$\text{and } \sqrt{c}=e+\frac{1}{e_1+}\,\frac{1}{e_2+}\,\frac{1}{e_3+}\ \&c.$$

(*B*. 143—8; *Legendre, Theor. Nomb.* 28—33.)

The periods in the series a_1, a_2, a_3, &c. e_1, e_2, e_3, &c. for all values of c from 1 to 100 will be found in the annexed table.

$\sqrt{c}$	$a_1\ a_2\ a_3$ &c.	$e_1\ e_2\ e_3$ &c.
$\sqrt{2}$	1	2
$\sqrt{3}$	2 1	1 2
$\sqrt{5}$	1	4
$\sqrt{6}$	2 1	2 4
$\sqrt{7}$	3 2 3 1	1 1 1 4
$\sqrt{8}$	4 1	1 4
$\sqrt{10}$	1	6
$\sqrt{11}$	2 1	3 6
$\sqrt{12}$	3 1	2 6
$\sqrt{13}$	4 3 3 4 1	1 1 1 1 6
$\sqrt{14}$	5 2 5 1	1 2 1 6
$\sqrt{15}$	6 1	1 6
$\sqrt{17}$	1	8
$\sqrt{18}$	2 1	4 8
$\sqrt{19}$	3 5 2 5 3 1	2 1 3 1 2 8
$\sqrt{20}$	4 1	2 8
$\sqrt{21}$	5 4 3 4 5 1	1 1 2 1 1 8
$\sqrt{22}$	6 3 2 3 6 1	1 2 4 2 1 8
$\sqrt{23}$	7 2 7 1	1 3 1 8
$\sqrt{24}$	8 1	1 8
$\sqrt{26}$	1	10
$\sqrt{27}$	2 1	5 10
$\sqrt{28}$	3 4 3 1	3 2 3 10
$\sqrt{29}$	4 5 5 4 1	2 1 1 2 10
$\sqrt{30}$	5 1	2 10
$\sqrt{31}$	6 5 3 2 3 5 6 1	1 1 3 5 3 1 1 10
$\sqrt{32}$	7 4 7 1	1 1 1 10
$\sqrt{33}$	8 3 8 1	1 2 1 10
$\sqrt{34}$	9 2 9 1	1 4 1 10
$\sqrt{35}$	10 1	1 10
$\sqrt{37}$	1	12
$\sqrt{38}$	2 1	6 12
$\sqrt{39}$	3 1	4 12
$\sqrt{40}$	4 1	3 12
$\sqrt{41}$	5 5 1	2 2 12
$\sqrt{42}$	6 1	2 12
$\sqrt{43}$	7 6 3 9 2 9 3 6 7 1	1 1 3 1 5 1 3 1 1 12
$\sqrt{44}$	8 5 7 4 7 5 8 1	1 1 1 2 1 1 1 12
$\sqrt{45}$	9 4 5 4 9 1	1 2 2 2 1 12
$\sqrt{46}$	10 3 7 6 5 2 5 6 7 3 10 1	1 3 1 1 2 6 2 1 1 3 1 12

$\sqrt{47}\begin{cases}11\ 2\ 11\ 1\\ 1\ 5\ 1\ 12\end{cases}$

$\sqrt{48}\begin{cases}12\ 1\\ 1\ 12\end{cases}$

$\sqrt{50}\begin{cases}1\\ 14\end{cases}$

$\sqrt{51}\begin{cases}2\ 1\\ 7\ 14\end{cases}$

$\sqrt{52}\begin{cases}3\ 9\ 4\ 9\ 3\ 1\\ 4\ 1\ 2\ 1\ 4\ 14\end{cases}$

$\sqrt{53}\begin{cases}4\ 7\ 7\ 4\ 1\\ 3\ 1\ 1\ 3\ 14\end{cases}$

$\sqrt{54}\begin{cases}5\ 9\ 2\ 9\ 5\ 1\\ 2\ 1\ 6\ 1\ 2\ 14\end{cases}$

$\sqrt{55}\begin{cases}6\ 5\ 6\ 1\\ 2\ 2\ 2\ 14\end{cases}$

$\sqrt{56}\begin{cases}7\ 1\\ 2\ 14\end{cases}$

$\sqrt{57}\begin{cases}8\ 7\ 3\ 7\ 8\ 1\\ 1\ 1\ 4\ 1\ 1\ 14\end{cases}$

$\sqrt{58}\begin{cases}9\ 6\ 7\ 7\ 6\ 9\ 1\\ 1\ 1\ 1\ 1\ 1\ 1\ 14\end{cases}$

$\sqrt{59}\begin{cases}10\ 5\ 2\ 5\ 10\ 1\\ 1\ 2\ 7\ 2\ 1\ 14\end{cases}$

$\sqrt{60}\begin{cases}11\ 4\ 11\ 1\\ 1\ 2\ 1\ 14\end{cases}$

$\sqrt{61}\begin{cases}12\ 3\ 4\ 9\ 5\ 5\ 9\ 4\ 3\ 12\ 1\\ 1\ 4\ 3\ 1\ 2\ 2\ 1\ 3\ 4\ 1\ 14\end{cases}$

$\sqrt{62}\begin{cases}13\ 2\ 13\ 1\\ 1\ 6\ 1\ 14\end{cases}$

$\sqrt{63}\begin{cases}14\ 1\\ 1\ 14\end{cases}$

$\sqrt{65}\begin{cases}1\\ 16\end{cases}$

$\sqrt{66}\begin{cases}2\ 1\\ 8\ 16\end{cases}$

$\sqrt{67}\begin{cases}3\ 6\ 7\ 9\ 2\ 9\ 7\ 6\ 3\ 1\\ 5\ 2\ 1\ 1\ 7\ 1\ 1\ 2\ 5\ 16\end{cases}$

$\sqrt{68}\begin{cases}4\ 1\\ 4\ 16\end{cases}$

$\sqrt{69}\begin{cases}5\ 4\ 11\ 3\ 11\ 4\ 5\ 1\\ 3\ 3\ 1\ 4\ 1\ 3\ 3\ 16\end{cases}$

$\sqrt{70}\begin{cases}6\ 9\ 5\ 9\ 6\ 1\\ 2\ 1\ 2\ 1\ 2\ 16\end{cases}$

$\sqrt{71}\begin{cases}7\ 5\ 11\ 2\ 11\ 5\ 7\ 1\\ 2\ 2\ 1\ 7\ 1\ 2\ 2\ 16\end{cases}$

$\sqrt{72}\begin{cases}8\ 1\\ 2\ 16\end{cases}$

$\sqrt{73}\begin{cases}9\ 8\ 3\ 3\ 8\ 9\ 1\\ 1\ 1\ 5\ 5\ 1\ 1\ 16\end{cases}$

$\sqrt{74}\begin{cases}10\ 7\ 7\ 10\ 1\\ 1\ 1\ 1\ 1\ 16\end{cases}$

$\sqrt{75}\begin{cases}11\ 6\ 11\ 1\\ 1\ 1\ 1\ 16\end{cases}$

$\sqrt{76}\begin{cases}12\ 5\ 8\ 9\ 3\ 4\ 3\ 9\ 8\ 5\ 12\ 1\\ 1\ 2\ 1\ 1\ 5\ 4\ 5\ 1\ 1\ 2\ 1\ 16\end{cases}$

$\sqrt{77}\begin{cases}13\ 4\ 7\ 4\ 13\ 1\\ 1\ 3\ 2\ 3\ 1\ 16\end{cases}$

$\sqrt{78}\begin{cases}14\ 3\ 14\ 1\\ 1\ 4\ 1\ 16\end{cases}$

$\sqrt{79}\begin{cases}15\ 2\ 15\ 1\\ 1\ 7\ 1\ 16\end{cases}$

$\sqrt{80}\begin{cases}16\ 1\\ 1\ 16\end{cases}$

$\sqrt{82}\begin{cases}1\\ 18\end{cases}$

$\sqrt{83}\begin{cases}2\ 1\\ 9\ 18\end{cases}$

$\sqrt{84}\begin{cases}3\ 1\\ 6\ 18\end{cases}$

$\sqrt{85}\begin{cases}4\ 9\ 9\ 4\ 1\\ 4\ 1\ 1\ 4\ 18\end{cases}$

$\sqrt{86}\begin{cases}5\ 10\ 7\ 11\ 2\ 11\ 7\ 10\ 5\ 1\\ 3\ 1\ 1\ 1\ 8\ 1\ 1\ 1\ 3\ 18\end{cases}$

$\sqrt{87}\begin{cases}6\ 1\\ 3\ 18\end{cases}$

$\sqrt{88}\begin{cases}7\ 9\ 8\ 9\ 7\ 1\\ 2\ 1\ 1\ 1\ 2\ 18\end{cases}$

$\sqrt{89}\begin{cases}8\ 5\ 5\ 8\ 1\\ 2\ 3\ 3\ 2\ 18\end{cases}$

$\sqrt{90}\begin{cases}9\ 1\\ 2\ 18\end{cases}$

$\sqrt{91}\begin{cases}10\ 9\ 3\ 14\ 3\ 9\ 10\ 1\\ 1\ 1\ 5\ 1\ 5\ 1\ 1\ 18\end{cases}$

$\sqrt{92}\begin{cases}11\ 8\ 7\ 4\ 7\ 8\ 11\ 1\\ 1\ 1\ 2\ 4\ 2\ 1\ 1\ 18\end{cases}$

$\sqrt{93}\begin{cases}12\ 7\ 11\ 4\ 3\ 4\ 11\ 7\ 12\ 1\\ 1\ 1\ 1\ 4\ 6\ 4\ 1\ 1\ 1\ 18\end{cases}$

$\sqrt{94}\begin{cases}13\ 6\ 5\ 9\ 10\ 3\ 15\ 2\ 15\ 3\ 10\ 9\ 5\ 6\ 13\ 1\\ 1\ 2\ 3\ 1\ 1\ 5\ 1\ 8\ 1\ 5\ 1\ 1\ 3\ 2\ 1\ 18\end{cases}$

$\sqrt{95}\begin{cases}14\ 5\ 14\ 1\\ 1\ 2\ 1\ 18\end{cases}$

$\sqrt{96}\begin{cases}15\ 4\ 15\ 1\\ 1\ 3\ 1\ 18\end{cases}$

$\sqrt{97}\begin{cases}16\ 3\ 11\ 8\ 9\ 9\ 8\ 11\ 3\ 16\ 1\\ 1\ 5\ 1\ 1\ 1\ 1\ 1\ 1\ 5\ 1\ 18\end{cases}$

$\sqrt{98}\begin{cases}17\ 2\ 17\ 1\\ 1\ 8\ 1\ 18\end{cases}$

$\sqrt{99}\begin{cases}18\ 1\\ 1\ 18\end{cases}$

(*E. Add.* 41.)

The last quotient in the period $=2e$, e being the nearest root. (*B*. 149.)

GENERAL PROPERTIES OF EQUATIONS.

(32.) Every equation of n dimensions in x may be reduced to the form

$$x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots\dots + a_1 x + a = 0;$$ [α]

which for convenience may be represented by $\phi(x) = 0$.

If $\phi(x)$ be divided by $x - a$, the remainder will be $\phi(a)$. [β]

If $\phi(x) \div x - a$, a is a root of $\phi(x) = 0$; and conversely.

Every equation has at least one root.

(*Du Bourguet, Ann. de Math. Tome* 2.)

Every equation of n dimensions has n roots, and no more.

If a_r is the r^{th} root of $\phi(x) = 0$, then

$$\phi(x) = (x - a_1)(x - a_2)\dots\dots(x - a_n).$$ [γ]

The coefficient of x^{m-1} in $\phi(x)$ is the sum of all the combinations of the quantities $-a_1$, $-a_2$, $-a_3$, ... &c. $-a_n$, taken $n - m + 1$ at a time: [δ] thus

---

[α] $$\overset{n+1}{\mathrm{S}}_m\, a_{n-m+1}\, x^{n-m+1} = 0,$$

or $$0 = \overset{n+1}{\mathrm{S}}_m\, a_{m-1}\, x^{m-1};$$

a_n being unity: this will frequently be found to be a more convenient mode of arrangement than the former.

[β] $$\frac{\overset{n+1}{\mathrm{S}}_m\, a_{m-1}\, x^{m-1}}{x - a} = \overset{n}{\mathrm{S}}_m\, x^{m-1} \overset{n-m+1}{\mathrm{S}}_r\, a_{m+r-1}\, a^{r-1} + \frac{\overset{n+1}{\mathrm{S}}_m\, a_{m-1}\, a^{m-1}}{x - a}.$$

[γ] $$\overset{n+1}{\mathrm{S}}_m\, a_{m-1}\, x^{m-1} = \overset{n}{\mathrm{P}}_r\,(x - a_r);$$

$$= \overset{n+1}{\mathrm{S}}_m\, x^{m-1} \overset{n-m+1,n}{\mathrm{C}}_r\,(-a_r).$$

[δ] $$a_{m-1} = \overset{n-m+1,n}{\mathrm{C}}_r\,(-a_r).$$

a_{n-1} = sum of the roots, with their signs changed;

a_{n-2} = sum of their products taken two at a time, with their signs changed;

&c. = &c.

a = product of all the roots, with their signs changed;

$\frac{a_{m-1}}{a}$ = sum of all the combinations of the quantities

$$-\frac{1}{a_1}, -\frac{1}{a_2}, -\frac{1}{a_3}, \text{\&c.} -\frac{1}{a_n},$$

taken $m-1$ at a time.

If $a_{n-m+1} = a_{m-1}$, then a_r being a root, $\frac{1}{a_r}$ is also a root.

If the coefficients are alternately positive and negative, the roots are all positive; if the coefficients are all positive, the roots are all negative.

If the signs of all the terms are changed, the roots are not changed.

If the signs of the alternate terms beginning with the second are changed, the signs of all the roots are changed.

If the coefficients are integers, the equation cannot have a fractional root.

If any coefficient be altered, all the roots are altered.

If n is odd, $\phi(x) = 0$ has at least one possible root, which has the same sign as $-a$.

If n is even, and a negative, the equation has two possible roots, one positive, the other negative.

$$\frac{a_{m-1}}{a} = \overset{m-1,n}{C_r}\left(-\frac{1}{a_r}\right).$$

If a is positive, there is an even, and if negative, an odd, number of positive possible roots.

If $\alpha + \sqrt{\beta}$ is a root of an equation of which the coefficients are rational, β not being a complete square, then $\alpha - \sqrt{\beta}$ is also a root.

(*W.* 280—97; *Bour.* 275—80; *L.* 178—94.)

TRANSFORMATION OF EQUATIONS.

(33.) To transform the equation $\phi(x) = 0$ into another whose roots are $a_1 - c$, $a_2 - c$, &c. $a_n - c$: assume $x = y + c$, and by substituting this value of x in $\phi(x) = 0$, we obtain

$$\begin{array}{l|l|l} y^n + nc & y^{n-1} + \dfrac{n(n-1)}{1\,.\,2}\,.\,c^2 & y^{n-2} + \&c. \\ + a_{n-1} & \quad + (n-1)\,a_{n-1}c & \\ & \quad + \qquad a_{n-2} & \end{array}$$

$$\begin{array}{l|l|l} + n \quad . \qquad c^{n-1} & y + \qquad c^n & = 0;\,[a] \\ + (n-1)a_{n-1}c^{n-2} & \quad + a_{n-1}c^{n-1} & \\ + (n-2)a_{n-2}c^{n-3} & \quad + a_{n-2}c^{n-2} & \\ + \&c. & \quad + \&c. & \end{array}$$

which may for convenience be thus expressed:

$$y^n + \frac{\psi^{n-1}(c)}{1.2\ldots(n-1)}\,y^{n-1} + \frac{\psi^{n-2}(c)}{1.2\ldots(n-2)}\,y^{n-2} + \&c.$$

$$+ \frac{\psi^2(c)}{1.2}\,y^2 + \frac{\psi(c)}{1}\,y + \phi(c) = 0;$$

$$\text{or} \quad \phi(y+c) = 0.$$

---

[a] $$0 = \overset{n+1}{\mathrm{S}}_m \frac{y^{m-1}}{\lfloor\underline{m-1}} \,.\, \overset{n-m+2}{\mathrm{S}}_r \,\underset{m-1}{\lfloor\underline{m+r-2}} \,.\, a_{m+r-2} \,.\, c^{r-1};$$

$$= \phi(c) + \overset{n}{\mathrm{S}}_m \frac{y^m}{\lfloor\underline{m}} \,.\, \psi^m(c).$$

To take away the m^{th} term of $\phi(x) = 0$: assume the coefficient of the m^{th} term of the above equation,

$$\frac{n(n-1)\dots(n-m+2)}{1.\ 2\ \dots\ (m-1)} c^{m-1} + \frac{(n-1)\dots(n-m+2)}{1\ \dots\ (m-2)} a_{n-1} c^{m-2} + \&\text{c.}$$

$$+ (n-m+2)\, a_{n-m+2}\, c + a_{n-m+1} = 0.$$ [β]

hence, to take away the m^{th} term we must solve an equation of $m-1$ dimensions.

$$\text{If } m = 2, \qquad nc + a_{n-1} = 0;$$

$$m = 3, \qquad \frac{n(n-1)}{1.\ 2} c^2 + (n-1)\, a_{n-1} c + a_{n-2} = 0.$$

To transform an equation into one whose roots are respectively m times the roots of the original equation: multiply the successive terms by $1, m, m^2$, &c.

If the coefficients be respectively divided by $1, m, m^2$, &c. the roots will all be divided by m.

To transform the equation $\phi(x) = 0$ into one whose roots are the reciprocals of the roots of the former: assume $x = \frac{1}{y}$, we obtain

$$y^n + \frac{a_1}{a} y^{n-1} + \dots + \frac{a_{n-2}}{a} y^2 + \frac{a_{n-1}}{a} y + \frac{1}{a} = 0.$$

(*W.* 280—97; *Bour.* 275—80; *G.* 310—25.)

[β] To take away the coefficient of y^{n-m+1}, assume

$$\overset{m}{\mathrm{S}}_r \frac{\lfloor\underline{n-m+r}_{\,r-1}}{\lfloor\underline{r-1}}\, a_{n-m+r}\, c^{r-1} = 0.$$

ELIMINATION.

(34.) If two equations each containing two unknown quantities, have a common divisor containing both unknown quantities, the number of solutions is infinite.

If they have a common divisor containing only one unknown quantity as x, x will have a finite, y an infinite number of values.

To eliminate one of the unknown quantities as x: arrange both equations according to the powers of x, and proceed as in finding the greatest common divisor; we shall at length obtain a remainder containing only y, which being made $=0$, will give the final equation required.

If this equation be capable of solution, the values of x may be found by substituting those of y obtained from it, in the preceding remainder.

If we have three equations (1), (2), (3), containing three unknown quantities, x may be eliminated between (1) and (2), and (1) and (3); and two equations thus obtained, each containing y and z, from which y may be eliminated; the result will be the final equation in terms of z only.

The same method may be applied to n equations, containing n unknown quantities.

(*Bour*. 281—90; *L*. 184—96.)

If b_1, b_2, &c. are the factors that have been introduced for the purpose of avoiding fractional quotients, the last remainder divided by $b_1 . b_2 \ldots$ &c. will be the true final equation.

In general, the degree of the final equation equals the number of separate systems of values that satisfy the given equations.

If the last remainder is independent of the unknown quantities, the equations are incompatible.

Further observations, and their application to particular cases. (*Bour.* 361—84.)

Another method. Let the equations be reduced to the form

$$1+A_1x+A_2x^2+A_3x^3+\ldots+A_mx^m=0, \qquad (1)$$

$$1+\frac{a_1}{x}+\frac{a_2}{x^2}+\frac{a_3}{x^3}+\ldots\frac{a_n}{x^n}=0, \qquad (2)$$

and let

$$(A_1+A_2x+A_3x^2+\ldots+A_nx^{n-1})^2={}^2A_1+{}^2A_2x+{}^2A_3x^2+\text{\&c.}$$

$$(A_1+A_2x+A_3x^2+\ldots+A_mx^{m-1})^3={}^3A_1+{}^3A_2x+{}^3A_3x^2+\text{\&c.}$$

$$\text{\&c.}=\text{\&c.}$$

$$\left(a_1+\frac{a_2}{x}+\frac{a_3}{x^2}+\ldots+\frac{a_n}{x^{n-1}}\right)^2={}^2a_1+\frac{{}^2a_2}{x}+\frac{{}^2a_3}{x^2}+\text{\&c.}$$

$$\left(a_1+\frac{a_2}{x}+\frac{a_3}{x^2}+\ldots+\frac{a_n}{x^{n-1}}\right)^3={}^3a_1+\frac{{}^3a_2}{x}+\frac{{}^3a_3}{x^2}+\text{\&c.}$$

$$\text{\&c.}=\text{\&c.}$$

also assume

$$B_1=A_1, \qquad =A_1;$$

$$B_2=A_2-\tfrac{1}{2}\,{}^2A_1, \qquad =\tfrac{1}{2}(2A_2-A_1.B_1);$$

$$B_3=A_3-\tfrac{1}{2}\,{}^2A_2+\tfrac{1}{3}\,{}^3A_1, \qquad =\tfrac{1}{3}(3A_3-2A_1.B_2-A_2.B_1);$$

$$B_4=A_4-\tfrac{1}{2}\,{}^2A_3+\tfrac{1}{3}\,{}^3A_2-\tfrac{1}{4}\,{}^4A_1, \qquad =\tfrac{1}{4}(4A_4-3A_1.B_3-2A_2.B_2-A_3.B_1);$$

$$\text{\&c.}=\text{\&c.} \qquad =\text{\&c.}$$

$$\begin{aligned} b_1 &= a_1, &&= a_1; \\ b_2 &= a_2 - \tfrac{1}{2}\,{}^2a_1, &&= \tfrac{1}{2}(2a_2 - a_1.b_1); \\ b_3 &= a_3 - \tfrac{1}{2}\,{}^2a_2 + \tfrac{1}{3}\,{}^3a_1, &&= \tfrac{1}{3}(3a_3 - 2a_1.b_2 - a_2.b_1); \\ \&c. &= \&c. &&= \&c. \end{aligned}$$

and let $c = B_1.b_1 + 2B_2.b_2 + 3B_3.b_3 +$ &c; then

$$1 - c + \frac{c^2}{1.2} - \frac{c^3}{1.2.3} + \&c. = 0, \text{ or } \epsilon^{-c} = 0$$

is the final equation required; from which however it is necessary to exclude all products of the quantities A_1, A_2, &c. exceeding m dimensions, and of a_1, a_2, &c. exceeding n dimensions.

(*G. A.* 44.)

Third method. If $ax^n = a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} +$ &c. multiplying both sides by ax, and substituting for ax^n its value, we obtain

$$\begin{aligned} a^2x^{n+1} = (a_1.a_1 + a.a_2)x^{n-1} &+ (a_1.a_2 + a.a_3)x^{n-2} \\ &+ (a_1.a_3 + a.a_4)x^{n-3} + \&c. \end{aligned}$$

assume $a_1.a_1 + a.a_2 = b_1$, $a_1.a_2 + a.a_3 = b_2$, &c.

again multiplying by ax, and substituting as before, we have

$$\begin{aligned} a^3x^{n+2} &= (a_1.b_1 + a.b_2)x^{n-1} + (a_1.b_2 + a.b_3)x^{n-2} + \&c. \\ &= c_1x^{n-1} + c_2x^{n-2} + \&c. \end{aligned}$$

and so on. We have

$$\begin{aligned} b_1 &= a_1.a_1 + a.a_2, \\ c_1 &= a_1.b_1 + a.a_2.a_1 + a^2.a_3, \\ d_1 &= a_1.c_1 + a.a_2.b_1 + a^2.a_3.a_1 + a^3.a_4, \\ e_1 &= a_1.d_1 + a.a_2.c_1 + a^2.a_3.b_1 + a^3.a_4.a_1 + a^4.a_5, \\ \&c. &= \&c. \end{aligned}$$

$b_2 = a_1 . a_2 + a . a_3,$

$c_2 = a_1 . b_2 + a . a_2 . a_2 + a^2 . a_4,$

$d_2 = a_1 . c_2 + a . a_2 . b_2 + a^2 . a_3 . a_2 + a^3 . a_5,$

&c. = &c.

If we have two equations of n, and $n+r$ dimensions in x respectively, we may, by substituting the values of x^{n+1}, x^{n+2}, &c. determined by this method, obtain an equation of $n-1$ dimensions, and from this last and the former, one of $n-2$ dimensions by the same method, and so on; we at length arrive at an equation of no dimensions in x, which is the final equation required. (*Kramp*, *Ann. de Math. Tome* 1.)

Application of the theory of symmetrical functions to elimination. (*L. C.* 10—4; *Waring*, *Medit. Alg.*)

DEPRESSION OF EQUATIONS.

(35.) *Equations containing equal roots.* If $\phi(x) = 0$ has m equal roots, $\phi'(x) = 0$ has $m-1$ of them: hence these two equations will have a common divisor ♃ $(x-a)^{m-1}$, which may be found, and the original equation thus reduced to another of $(n-m)$ dimensions. Also $\phi''(x) = 0$ has $m-2$ of the equal roots, and therefore $\phi(x) = 0$, $\phi'(x) = 0$, $\phi''(x) = 0$, will have a common divisor ♃ $(x-a)^{m-2}$.

If $\phi(x) = 0$ has two roots ♃ $\pm a$, they may be found by changing the signs of all the roots, and finding the greatest common divisor of the original and resulting equation.

(*W.* 319, 20; *Bour.* 292—8; *L.* 205—8.)

(36.) *Recurring equations.* Every recurring equation of $2n+1$ dimensions has one root $= \mp 1$, according as the last term is negative or positive, and may be reduced to an equation of $2n$ dimensions by division. (*W.* 295.)

The roots of a recurring equation of $2n$ dimensions may be found by the solution of an equation of n dimensions, if $n > 1$. (*W.* 325; *Bour.* 299—300.)

SYMMETRICAL FUNCTIONS OF THE ROOTS.

(37.) *Sums of the powers of the roots.* Let S_1, S_2, S_3, &c. represent the sums of the 1st, 2nd, 3rd, &c. powers of the roots.

$S_1 = -a_{n-1}$;

$S_2 = -a_{n-1} S_1 - 2a_{n-2}$;

$S_3 = -a_{n-1} S_2 - a_{n-2} S_1 - 3a_{n-3}$;

&c. = &c.

$S_m = -a_{n-1} S_{m-1} - a_{n-2} S_{m-2} - \dots - m a_{n-m}$; [α]

$S_{n+m} = -a_{n-1} S_{n+m-1} - a_{n-2} S_{n+m-2} - \dots - a S_m$. [β]

Let S_{-1}, S_{-2}, &c. represent the sums of the negative powers of the roots;

$S_{-1} = -\frac{a_1}{a}$;

$S_{-2} = -\frac{a_1}{a} S_{-1} - \frac{2}{a} a_2$;

$S_{-3} = -\frac{a_1}{a} S_{-2} - \frac{a_2}{a} S_{-1} - \frac{3}{a} a_3$;

$S_{-4} = -\frac{a_1}{a} S_{-3} - \frac{a_2}{a} S_{-2} - \frac{a_3}{a} S_{-1} - \frac{4}{a} a_4$; [γ]

&c. = &c.

$\frac{S_m}{S_{m-1}}$ continually approximates to the greatest, and $\frac{S_{-m+1}}{S_{-m}}$

[α] $-m a_{n-m} = \overset{m}{\mathrm{S}}_s a_{n-s+1} \cdot \overset{n}{\mathrm{S}}_r a_r^{m-s+1}$.

[β] $0 = \overset{n+1}{\mathrm{S}}_s a_{s-1} \cdot \overset{n}{\mathrm{S}}_r a_r^{m+s-1}$.

[γ] $-m a_m = \overset{m}{\mathrm{S}}_r a_{r-1} \cdot \overset{n}{\mathrm{S}}_s a_s^{-m+r-1}$.

to the least root, as m increases, if all the roots are possible.

(*W.* 352, 3; *Bour.* 302—7; *G. A.* Chap. v; *Gergonne, Ann. de Math. Tome.* 3.)

To express the sums of the powers of the roots in terms of the coefficients only, and conversely.

$$S_1 = -a_{n-1};$$

$$S_2 = -\frac{2}{1}a_{n-2} + \frac{2}{2}a^2_{n-1};$$

$$S_3 = -\frac{3}{1}a_{n-3} + \frac{3}{2}.2a_{n-1}.a_{n-2} - \frac{3}{3}a^3_{n-1};$$

$$S_4 = -\frac{4}{1}a_{n-4} + \frac{4}{2}(2a_{n-1}.a_{n-3} + a^2_{n-2}) - \frac{4}{3}3a^2_{n-1}a_{n-2} + \frac{4}{4}a^4_{n-1};\ ^{\delta}$$

&c. = &c.

(*Waring, Med. Alg.* Cap. i; *Arbogast, Calc. des Deriv.* 68–78.)

(38.) Let $T(a_1^{r_1}.a_2^{r_2}\ldots.a_m^{r_m})$ represent the sum of all possible transpositions of the products of the roots taken m at a time, the several roots in each transposition being raised to the r_1th, r_2th, &c, r_mth powers respectively.

$$S_{r_1}.S_{r_2} = S_{r_1+r_2} + T(a_1^{r_1}.a_2^{r_2});$$

$$S_{r_3}.T(a_1^{r_1}.a_2^{r_2}) = T(a_1^{r_1+r_3}.a_2^{r_2}) + T(a_1^{r_1}.a_2^{r_2+r_3}) + T(a_1^{r_1}.a_2^{r_2}.a_3^{r_3});$$

$$S_{r_4}.T(a_1^{r_1}.a_2^{r_2}.a_3^{r_3}) = T(a_1^{r_1+r_4}.a_2^{r_2}.a_3^{r_3}) + T(a_1^{r_1}.a_2^{r_2+r_4}.a_3^{r_3}) + T(a_1^{r_1}.a_2^{r_2}.a_3^{r_3+r_4}) + T(a_1^{r_1}.a_2^{r_2}.a_3^{r_3}.a_4^{r_4});$$

&c. = &c;

[δ] If a_r is the rth root of the equation $0 = \overset{n+1}{\mathrm{S}}_m c_{m-1}x^{n-m+1}$,

$$\text{then } \overset{n}{\mathrm{S}}_r a_r^m = m\overset{m}{\mathrm{S}}_r(-1)^{m-r+1}.\frac{\varpi^{r-1}.c_1^{m-r+1}}{m-r+1},$$

$$\text{and } c_m = \overset{m}{\mathrm{S}}_r\frac{\varpi^{r-1}e_1^{m-r+1}}{\underline{|m-r+1}},\ \left\{e_m = -\frac{1}{m}.\overset{n}{\mathrm{S}}_r a_r^m\right\}.$$

$$S_{r_m}\cdot T(a_1^{r_1}\cdot a_2^{r_2}\cdot a_3^{r_3}\ldots a_{m-1}^{r_{m-1}}) = T(a_1^{r_1+r_m}\cdot a_2^{r_2}\cdot a_3^{r_3}\ldots a_{m-1}^{r_{m-1}})$$

$$+\ T(a_1^{r_1}\cdot a^{r_2+r_m}\cdot a_3^{r_3}\ldots a_{m-1}^{r_{m-1}}) + \text{\&c.}$$

$$+\ T(a_1^{r_1}\cdot a_2^{r_2}\cdot a_3^{r_3}\ldots a_{m-1}^{r_{m-1}+r_m})$$

$$+\ T(a_1^{r_1}\cdot a_2^{r_2}\cdot a_3^{r_3}\ldots a_m^{r_m}).$$

Hence $T(a_1^{r_1}\cdot a_2^{r_2}) = S_{r_1}\cdot S_{r_2} - S_{r_1+r_2}.$

If $r_1 = r_2$, $T(\left.\overline{a_1\cdot a_2}\right|^{r_1}) = \frac{1}{2}(\left.\overline{S_{r_1}}\right|^2 - S_{2r_1}).$

$$T(a_1^{r_1}\cdot a_2^{r_2}\cdot a_3^{r_3})$$

$$= S_{r_1}\cdot S_{r_2}\cdot S_{r_3} - S_{r_1+r_2}\cdot S_{r_3} - S_{r_1+r_3}\cdot S_{r_2} - S_{r_1}\cdot S_{r_2+r_3} + 2S_{r_1+r_2+r_3}.$$

If $r_2 = r_3$,

$$T(a_1^{r_1}\cdot\left.\overline{a_2\cdot a_3}\right|^{r_2}) = \tfrac{1}{2}(S_{r_1}\cdot\left.\overline{S_{r_2}}\right|^2 - S_{r_1}\cdot S_{2r_2}) - S_{r_1+r_2}\cdot S_{r_2} + S_{r_1+2r_2}.$$

If $r_1 = r_2 = r_3$,

$$T(\left.\overline{a_1\cdot a_2\cdot a_3}\right|^{r_1}) = \tfrac{1}{6}\left.\overline{S_{r_1}}\right|^3 - \tfrac{1}{2}S_{r_1}\cdot S_{2r_1} + \tfrac{1}{3}S_{3r_1}.$$

Every symmetrical function of the roots of $\phi(x) = 0$, and the coefficients of every equation whose roots are symmetrical functions of the roots of that equation, may be expressed in terms of a_1, a_2, &c. the coefficients of $\phi(x) = 0$.

(*G. A.* Chap. vii; *Bour.* 303, 8.)

THE EQUATION OF DIFFERENCES.

(39.) Let $y^r + b_{r-1}y^{r-1} + b_{r-2}y^{r-2} + \ldots + b_1 y + b = 0$ be the equation whose roots are the squares of the differences of the roots of $\phi(x) = 0$, and σ_1, σ_2, &c. the sums of the 1st, 2nd, &c. powers of its roots.

$$\sigma_m = S_0\cdot S_{2m} - \frac{2m}{1}S_1\cdot S_{2m-1} + \frac{2m(2m-1)}{1\ .\ 2}S_2\cdot S_{2m-2}$$

$$- \frac{2m(2m-1)(2m-2)}{1\,.\,2\,.\,3} S_3 . S_{2m-3} + \&c.$$

$$+ (-1)^m . \frac{2m(2m-1)\ldots(m+1)}{1\,.\,2\,\ldots\,m} . \tfrac{1}{2}(S_m)^2.$$

$b_{r-1} = -\sigma_1$;

$b_{r-2} = -\frac{1}{2}(b_{r-1}\sigma_1 + \sigma_2)$;

$b_{r-3} = -\frac{1}{3}(b_{r-2}\sigma_1 + b_{r-1}\sigma_2 + \sigma_3)$;

$b_{r-4} = -\frac{1}{4}(b_{r-3}\sigma_1 + b_{r-2}\sigma_2 + b_{r-1}\sigma_3 + \sigma_4)$;

&c. = &c.

$$r = \frac{n(n-1)}{1\,.\,2}.$$

If the given equation is a cubic, the transformed equation is also a cubic, and

$b_2 = 2(3a_1 - a_2^2)$;

$b_1 = (3a_1 - a_2^2)^2$;

$b = -\frac{1}{3}\{4(3a_1 - a_2^2)(a_1^2 - 3a_2.a) + (9a - a_1a_2)^2\}$.

If the given equation is a biquadratic, the transformed equation is of six dimensions, and

$b_5 = 8a_2$;

$b_4 = 22a_2^2 + 8a$;

$b_3 = 18a_2^3 - 16a_2.a - 26a_1^2$;

$b_2 = 17a_2^4 + 24a_2^2.a - 7.2^4.a^2 + 3.2^4.a_2.a_1^2$;

$b_1 = 4a_2^5 + 2.3^3.a_2^2.a_1^2 + 2^3.3^3.a_1^2.a - 3.2^6.a_2.a^2 + 2^5.a_2^3.a$;

$b = 2^8.a^3 - 2^7.a_2^2.a^2 + 2^4.3^2.a_2.a_1^2.a + 2^4.a_2^4.a - 2^2.a_2^3.a_1^2 - 3^3.a_1^4$.

The biquadratic is here supposed to have been deprived of its 2nd term.

If b is negative the biquadratic will have two possible, and two impossible roots: if b is positive, the roots are all possible, when b_2 and b_4 are positive, and b_1, b_3, b_5, negative; otherwise they are all impossible.

To obtain the equation of differences by elimination: for x substitute $x+z$ in $\phi(x)=0$, and in after suppressing $\phi(x)$ in the transformed equation, (*Art.* 33) eliminate x between that equation, and $\phi(x)=0$. If y be substituted for z^2 in the resulting equation, which contains only the even powers of z, it will appear under the same form as above.

(*G.* 369—9; *G. A.* Chap. vi; *Bour.* 310, 1.)

LIMITS OF THE ROOTS OF EQUATIONS.

(40.) The limits of the roots of an equation, if substituted successively for the unknown quantity, give results alternately positive, and negative.

If two quantities when substituted in an equation give results having different signs, an odd number of roots must lie between them: if the results have the same sign, an even number of roots, or none, must lie between the quantities substituted.

To find a superior limit to the roots of an equation; in the transformed equation, $\phi(y+c)=0$, assume c such that all the coefficients may be positive; this value of c will be a superior limit.

To find an inferior limit, change the signs of all the roots, and proceed as before.

The greatest negative coefficient increased by unity is a superior limit.

If a_{n-p} is the first, and a_{n-r} the greatest negative coefficient, then

$(a_{n-r})^{\frac{1}{p}}+1$ is a superior limit.

(*Maiziere, Ann. de Math. Tome* 3.)

A negative coefficient less than the greatest increased by unity will frequently be found to be a superior limit: let a_r be the greatest negative coefficient, and a_{r+1} positive, then since

$$a_{r+1}x^{r+1}-a_rx^r=a_{r+1}x^r\left(x-\frac{a_r}{a_{r+1}}\right),$$

this part of the equation is essentially positive for all values of $x>\dfrac{a_r}{a_{r+1}}$, therefore the next greatest negative coefficient $+1$, if $>\dfrac{a_r}{a_{r+1}}$, is a superior limit.

If the roots are all possible, and positive, each of the following quantities is a superior limit;

$$\overline{\frac{1\,.\,2}{n(n-1)}a_{n-2}}\Bigg|^{\frac{1}{2}}, \qquad \overline{\frac{a^2_{n-1}-2a_{n-2}}{n}}\Bigg|^{\frac{1}{2}},$$

$$\overline{\frac{1\,.\,2}{n(n-1)}(a^2_{n-2}-2a_{n-1}.a_{n-3}+2a_{n-4})}\Bigg|^{\frac{1}{2}}. \qquad (W.\ 305.)$$

The roots of the equation $\phi'(x)=0$ are limits between those of the equation $\phi(x)=0$, when the roots of the latter are possible.

If all the roots of an equation are positive, or all negative and its terms be multiplied by the terms of any arithmetical progression, the resulting equation will be a limiting equation to the former,

If the equation has both positive and negative roots, the same is true; with this exception, that either two of its roots or none lie between the positive and negative roots of the original equation, according as a decreasing or an increasing progression is used.

(*W*. 298—318; *Bour*. 318—27; *F*. 507—11.)

Descartes's Rule. An equation cannot have a greater number of positive roots, than it has changes of sign from term to term, nor a greater number of negative roots than it has continuations of the same sign.

If the roots are all possible, there are exactly as many of them positive as there are changes, and as many negative as there are continuations of sign.

(*Bour.* 328—30; *G. Chap.* 34.)

INVESTIGATION OF COMMENSURABLE ROOTS.

(41.) Let a be a commensurable root of $\phi(x)=0$, and therefore an integer; and assume

$$\frac{a}{a}=q_1;$$

$$\frac{q_1+a_1}{a}=q_2;$$

$$\&c. = \&c.$$

$$\frac{q_{n-1}+a_{n-1}}{a}=-a_n;$$

q_1, q_2, &c. q_{n-1}; are all integers. Hence, to find the commensurable roots: write down in a horizontal line all the divisors of the last term, negative as well as positive, that are between the nearest assignable limits of the greatest positive, and negative roots; under these write the corresponding quotients of the last term divided by each of them; add to each of these the coefficient of x, and write down the sum under the corresponding quotient; then divide each of these sums by the divisor standing over it, and write down the quotient under it, neglecting all fractional quotients; to the remaining quotients add the coefficient of x^2, and proceed as before; and so on: those divisors which give always an integral quotient, will be found to be roots of the equation; which may therefore be reduced as many degrees.

If there are equal roots, this method will only detect one of them, it should therefore be repeated with the resulting equation.

We need not include 1 or -1 amongst the divisors, as it is easy to ascertain whether either is a root of the equation.

The number of divisors to which the above process is applied may be thus reduced. Let b_1, b_2 be the results obtained by substituting $+1$ and -1, in the equation:

[1] Every positive divisor which when diminished by unity does not divide b_1, and when increased by unity, does not divide b_2, may be rejected.

[2] Every negative divisor which when increased by unity does not divide b_1, and when diminished by unity does not divide b_2, may also be rejected.

(*Bour.* 331—7; *L.* 199—203.)

If three, or more terms of the arithmetical progression 1, 0, -1, &c. be substituted for the unknown quantity, and the divisors of the results, taken in order, be formed into arithmetical progressions, those divisors of the last term which occur in these progressions *may* be found on trial to be roots of the equation. (*W.* 336—40.)

INVESTIGATION OF INCOMMENSURABLE ROOTS.

(42.) *Case* 1st. Suppose the difference between any two possible roots > 1.

[1] *Lagrange's Method.* Find a number b_1 such that one of the roots is $> b_1$, but $< b_1 + 1$, and for x substitute $b_1 + \frac{1}{x_1}$ in the given equation. The resulting equation in x_1 must have at least one possible root > 1, let this root be $> b_2$ and $< b_2 + 1$;

for x_1 substitute $b_2 + \frac{1}{x_2}$, and repeat the same process with the resulting equation in x_2: and so on, as far as may be thought necessary. Then

$$x = b_1 + \frac{1}{b_2 +}\,\frac{1}{b_3 +}\,\frac{1}{b_4 +}\ \&c.$$

[2] *Newton's Method.* Determine by experiment a number c_1 such that one of the roots is $> c_1$, but $> c_1 + 0{,}1$; for x substitute $c_1 + y$, and from the resulting equation in y, obtain an approximate value of y, by neglecting y^2 and all superior powers: thus an approximate value of x is obtained. If greater accuracy is required, the same process may be repeated.

(*Bour.* 341—9; *Lagr. Equ. Num. Chap.* 3.)

Case 2nd. Suppose the difference between two possible roots to be < 1.

Determine the inferior limit $\frac{1}{l}$ of the positive roots of the equation of differences, and let k be the least integer $> \sqrt{l}$: transform the given equation into one whose roots are k times as great, in which equation if the series of natural numbers be successively substituted for x, one root and only one will lie between any two numbers m, and $m + 1$, and therefore the corresponding root of the given equation will lie between $\frac{k}{m}$, and $\frac{k}{m+1}$. A nearer approximation may be made to this root, by either of the preceding methods.

The equation must be cleared of equal roots, before this method can be applied.

(*Bour.* 350—7; *L* 217—20.)

QUADRATIC FACTORS.

(43.) Every equation of n dimensions has at least one possible quadratic factor, if every equation of $\frac{1}{2}n(n-1)$ dimensions has one possible factor either of the first or second degree.

Hence every equation of $2m$ dimensions may be decomposed into m quadratic factors. (*L. C.* 38, 9.)

To find the quadratic factors of $\phi(x)=0$ divide $\phi(x)$ by x^2+px+q, and continue the division, until the remainder contains only the first power of x, and may be represented by

$$f_1(p, q)x + f_2(p, q) = 0,$$

$$\text{then } f_1(p, q) = 0, \text{ and } f_2(p, q) = 0;$$

from which equations q may be eliminated, and the corresponding possible values of p and q determined as above (Art. 34.) will give the quadratic factors required. (*Bour.* 338, 9.)

Further investigation of the commensurable divisors of literal and numerical equations. (*Clairaut, Elém. d'Alg. P.* 3.)

IMPOSSIBLE ROOTS.

(44.) If $\alpha+\beta\sqrt{-1}$ is a root of an equation of which the coefficients are possible quantities, α and β being possible, $\alpha-\beta\sqrt{-1}$ is also a root.

Impossible quantities of all degrees may be reduced to the form $A+B\sqrt{-1}$, A and B being possible quantities.

To find the impossible roots: substitute $y+z\sqrt{-1}$ for x in the given equation, and another equation will be obtained

[illegible] $A+B\sqrt{-1}=0.$

Determine all the corresponding values of y and z that satisfy the equations $A=0$, $B=0$; let these be α_1, α_2, &c. β_1, β_2, &c. then $\alpha_1+\beta_1\sqrt{-1}$, $\alpha_2+\beta_2\sqrt{-1}$, &c. are the required roots.

The series of quantities, $-4\beta_1^2$, $-4\beta_2^2$, &c. will all be found amongst the negative roots of the equation of differences: to determine whether $-c$, a negative root of that equation, is one of these quantities, substitute $\frac{1}{2}\sqrt{c}$ for z in A and B, and if it be such, the resulting polynomials in y will have a common divisor.

If the equation of differences has other negative roots besides the above series, it must also have equal roots, which may be determined by preceding methods.

If any series of quantities be successively substituted for x, there can be only as many changes of sign in the results, as the equation has possible roots.

The equation $\phi'(x)=0$, has at least as many possible roots as $\phi(x)=0$, wanting one.

(*W.* 355—62; *G. A. Ch.* 2; *Bour.* 385—92; *F.* 533—5.)

Newton's rule for discovering impossible roots. (*W.* 363.)

APPLICATION OF THE THEORY OF EQUATIONS TO SURDS.

(45.) To extract the cube root of $a+\sqrt{b}$: assume

$$a+\sqrt{b}=z(x+\sqrt{y})^3,$$

then $x^2-y=\frac{1}{z}\cdot\overline{(a^2-b)z}|^{\frac{1}{3}}$;

z must be so assumed that $(a^2-b)z=c^3$, c being an integer.

Eliminating y we obtain

$$4zx^3-3czx-a=0;$$

which equation must have at least one possible root, in order that we may have

$$a+\sqrt{b}=z(x+\sqrt{y})^3.$$

To extract the n^{th} root of $a+\sqrt{b}$: assume

$$a+\sqrt{b}=z(x+\sqrt{y})^n;$$

then $x^2 - y = \frac{1}{z}\,\overline{(a^2-b)z^{n-2}}\Big|^{\frac{1}{n}}$.

z must be so assumed that $(a^2-b)z^{n-2}=c^n$, c being an integer.

Eliminating y between the equations

$$y = x^2 - c, \text{ and}$$

$$a = z\left(x^n + \frac{n(n-1)}{1.\ 2}x^{n-2}y + \&c.\right)$$

we obtain a final equation in x, which must have at least one possible root in order that we may have

$$a + \sqrt{b} = z(x+\sqrt{y})^n.$$

(*Bour.* 403—5; *L. C.* 47—9; *G. A. Chap.* 15.)

Cubic Equations.

(46.) *Cardan's Method.* Let the equation be reduced to the form

$$x^3 + a_1 x + a = 0.$$

Assume $y + z = x$, and $yz = -\frac{1}{3}a_1$

then $y = \left\{-\frac{a}{2} + \overline{\frac{a^2}{4} + \frac{a_1^3}{27}}\Big|^{\frac{1}{2}}\right\}^{\frac{1}{3}}$

$z = \left\{-\frac{a}{2} - \overline{\frac{a^2}{4} + \frac{a_1^3}{27}}\Big|^{\frac{1}{2}}\right\}^{\frac{1}{3}}$.

The three values of x are

$$y + z,$$

$$-\tfrac{1}{2}\{(y+z) + (y-z)\sqrt{-3}\},$$

$$-\tfrac{1}{2}\{(y+z) - (y-z)\sqrt{-3}\}.$$

If $\frac{a^2}{4} + \frac{a_1^3}{27} > 0$, y and z are possible, and a result may be obtained. In this case the equation has one possible, and two impossible roots.

If $\frac{a^2}{4} + \frac{a_1^3}{27} = 0$, the roots are $\left(\frac{r}{2}\right)^{\frac{1}{3}}$, $\left(\frac{r}{2}\right)^{\frac{1}{3}}$, and $-2\left(\frac{r}{2}\right)^{\frac{1}{3}}$.

If $\frac{a^2}{4} + \frac{a_1^3}{27} < 0$, y and z are impossible, and no result can be obtained. In this case all the roots are possible.

Let the equation be

$$x^3 - a_1 x - a = 0,$$

which has at least one possible root; let this be α, the other two roots are

$$-\frac{1}{2}(\alpha \pm \overline{4a_1 - 3\alpha^2}|^{\frac{1}{2}})$$

from which their values may be obtained, if the value of α be determined by approximation.

(*W.* 325—31; *E.* 734—49; *Bour.* 406—11; *G. A.* 58.)

Solution of a cubic equation by the method of divisors.

(*E.* 719—33.)

Solution by the theory of symmetrical functions.

(*Bour.* 415; *G. A.* 52.)

BIQUADRATIC EQUATIONS.

(47.) [1] *Descartes's Method.* Let the equation be reduced to the form

$$x^4 + a_2 x^2 + a_1 x + a = 0;$$

the first side of this equation may be supposed to be the product of two quadratic factors

$$x^2 + ex + f, \text{ and } x^2 - ex + g.$$

Multiplying these factors together, and equating the coefficients of x,

$$g + f - e^2 = a_2,$$

$$e(g - f) = a_1,$$

$$fg = a;$$

eliminating f, g, and putting y for e^2, we obtain

$$y^3 + 2a_2 y^2 + (a_2^2 - 4a)y - a_1^2 = 0.$$

This equation may be solved by either of the preceding methods, if two of its roots are impossible, and the *possible* value of y, or e^2 obtained; the values of x are

$$x = -\frac{e}{4} \pm \left(-\frac{a_2}{2} - \frac{e^2}{4} + \frac{a_1}{e}\right)^{\frac{1}{2}},$$

$$x = \frac{e}{4} \pm \left(-\frac{a_2}{2} - \frac{e^2}{4} - \frac{a_1}{e}\right)^{\frac{1}{2}}. \qquad (W.\ 332\text{—}3.)$$

The same reducing cubic equation may be obtained by substituting $y + z$ for x, and in the resulting equation putting

$$4zy^3 + (4z^2 + 2a_2 z + a_1)y = 0. \qquad (F.\ 551\text{—}2.)$$

[2] *Waring's Method.* Let the equation be

$$x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a = 0,$$

or $$x^4 + a_3 x^3 = -a_2 x^2 - a_1 x - a.$$

adding $\left(\frac{a_3^2}{4} + y\right)x^2 + \frac{a_3}{2} y \,.\, x + \frac{y^2}{4}$ to both sides of this equation, the second side becomes a perfect square, if

$$y^3 - a_2 y^2 + (a_1 \,.\, a_3 - 4a)y - a(a_3^2 - 4a_2) - a_1^2 = 0.$$

By substituting the possible value of y obtained from this equation, the preceding is reduced to two quadratics, whence the roots may be found. (*W.* 334, 5.)

This and the preceding method are applicable only when the proposed equation has two possible, and two impossible, roots.

[3] *Euler's Method.* Let the equation be reduced to the form

$$x^4 + a_2 x^2 + a_1 x + a = 0.$$

Assume $x = \sqrt{\beta_1} + \sqrt{\beta_2} + \sqrt{\beta_3}$, (1)
β_1, β_2, β_3, being the roots of the equation

$$y^3 + b_2 y^2 + b_1 y + b = 0.$$

Raising (1) to the 4th power, we have

$$x^4 + 2b_2 x^2 - 8(-b)^{\frac{1}{2}} x + b_2^2 - b_1 = 0;$$

equating the coefficients of x we obtain

$$b_2 = \frac{a_2}{2},$$

$$b_1 = \frac{a_2^2 - 4a}{16},$$

$$b = -\frac{a_1^2}{64}.$$

If a_1 is positive, the values of x are

$$x = +\sqrt{\beta_1} + \sqrt{\beta_2} - \sqrt{\beta_3},$$
$$x = +\sqrt{\beta_1} - \sqrt{\beta_2} + \sqrt{\beta_3},$$
$$x = -\sqrt{\beta_1} + \sqrt{\beta_2} + \sqrt{\beta_3},$$
$$x = -\sqrt{\beta_1} - \sqrt{\beta_2} - \sqrt{\beta_3}.$$

If a_1 is negative, the values of x are

$$x = +\sqrt{\beta_1} + \sqrt{\beta_2} + \sqrt{\beta_3},$$
$$x = +\sqrt{\beta_1} - \sqrt{\beta_2} - \sqrt{\beta_3},$$
$$x = -\sqrt{\beta_1} + \sqrt{\beta_2} - \sqrt{\beta_3},$$
$$x = -\sqrt{\beta_1} - \sqrt{\beta_2} + \sqrt{\beta_3}.$$ (*E.* 773—83.)

The same result may be obtained by assuming

$$x = u + y + z, \text{ and}$$

$$8uyz + a_1 = 0,$$

$$4(u^2 + y^2 + z^2) + 2a_2 = 0.$$

(*G. A.* 58; *Bour.* 412, 3.)

Solution of a biquadratic equation by the method of divisors.
(*E*. 759.)

Solution by the theory of symmetrical functions.
(*Bour*. 416, 7; *G. A*. 53, 5.)

THE EQUATION $y^n \pm p = 0$.

(48.) *Properties of the equation* $y^n \pm p = 0$. This equation may be reduced to $x^n \pm 1 = 0$, by assuming $y = p^{\frac{1}{n}} x$.

$x^{2m-1} - 1 = 0$ has only one possible root, 1.

$x^{2m} - 1 = 0$ has only two possible roots, 1 and -1.

$x^{2m-1} + 1 = 0$ has only one possible root, -1.

$x^{2m} + 1 = 0$ has no possible root.

$x^n \pm 1$ has no equal roots.

If a is a root of the equation $x^n - 1 = 0$, a^r is also a root, r being any integer. And $a^r = a^{r-n}$.

If a is a root of the equation $x^n + 1 = 0$, a^{2r-1} is also a root.

$S_{\pm m} = 0$, unless $m \mathrel{\subset\!\div\!\supset} n$, in which case $S_{\pm m} = n$.

If $n \mathrel{\subset\!\nmid\!\supset} n_1^{p_1} . n_2^{p_2} . n_3^{p_3} \ldots$ &c. n_1, n_2, &c. being prime numbers, the solution of the equation $x^n - 1 = 0$ may be made to depend on the solution of the equations $x^{n_1} - 1 = 0$, $x^{n_2} - 1 = 0$; &c.
(*Bour*. 393—401; *L. C*. 15—8; *G. A*. 46, 7.)

(49.) *Theorem of Fermat*. If p is a prime number, and a prime to p, then

$$a^{p-1} - 1 \mathrel{\subset\!\div\!\supset} p.$$

If a is such a number that no power of $a < a^{p-1} \mathrel{\subset\!\div\!\supset} p$, and a, a^2, $a^3, \ldots a^{p-1}$ be respectively divided by p, the remainders will all be unequal.

The values of a for all values of p from 3 to 37 inclusive will be found in the annexed Table.

p	a											
3	1											
5	2	3										
7	3	5										
11	2	6	7	8								
13	2	6	7	11								
17	3	5	6	7	11	12	14					
19	2	3	10	13	14	15						
23	5	7	10	11	13	14	15	17	20	21		
29	2	3	8	10	11	14	15	18	19	21	26	27
31	3	11	12	13	17	21	22	24				
37	2	5	13	15	17	18	19	20	22	24	32	35

(*G. A.* 48, 49; *Mem. Acad. Berlin, Ann.* 1771.)

(50.) *Algebraical solution of the equation* $x^p - 1 = 0$, p *being a prime number.*

The roots of this equation,

$$a,\ a^2,\ a^3,\ a^4,\ \text{\&c.}\ a^{p-1}$$

may be more conveniently represented by

$$a,\ a^a,\ a^{a^2},\ a^{a^3},\ \text{\&c.}\ a^{a^{p-2}},$$

in which a may have any value corresponding to the value of p in the preceding Table.

If a^a be substituted for a, the roots become

$$a^a,\ a^{a^2},\ a^{a^3},\ a^{a^4},\ \text{\&c.}\ a^{a^{p-2}},\ a^{a^{p-1}};$$

where $\alpha^{a^{p-1}} = a$, ∵ $a^{p-1} - 1 \div p$.

In this case the roots are evidently the same as before, but arranged in a different order.

Let β, β^2, β^3, β^{p-2}, &c. be the roots of the equation, $x^{p-1} - 1 = 0$; assume

$$c = \alpha + \beta\alpha^a + \beta^2\alpha^{a^2} + \ldots + \beta^{p-2}\alpha^{a^{p-2}};$$

then $c^{p-1} = (\alpha + \beta\alpha^a + \beta^2\alpha^{a^2} + \ldots + \beta^{p-2}\alpha^{a^{p-2}})^{p-1}$,

$$= A + A_1\beta + A_2\beta^2 + \ldots + A_{p-2}\beta^{p-2}:$$

observing that all indices of $\alpha > a^{p-1}$, and of $\beta > p-1$ may be respectively divided by those quantities, and the quotients neglected.

Each of the quantities A, A_1, A_2, &c. may be reduced to the form

$$B + C(\alpha + \alpha^a + \alpha^{a^2} + \ldots + \alpha^{a^{p-2}}),$$

in which B and C are known quantities independent of α; hence the values of A, A_1, &c. are not altered by substituting successively α^a, &c. in the place of α.

Let c_1^{p-1}, c_2^{p-1}, &c. c_{p-2}^{p-1}, be the values of c^{p-1}, when β^2, β^3, &c. β^{p-1}, are substituted for β, then

$$\alpha = \frac{1}{p-1}(\beta^{p-1}c + \beta^{2(p-1)}c_1 + \ldots + \beta^{(p-1)(p-1)}c_{p-2}),$$

$$\alpha^a = \frac{1}{p-1}(\beta^{p-2}c + \beta^{2(p-2)}c_1 + \ldots + \beta^{(p-1)(p-2)}c_{p-2}),$$

$$\alpha^{a^2} = \frac{1}{p-1}(\beta^{p-3}c + \beta^{2(p-3)}c_1 + \ldots + \beta^{(p-1)(p-3)}c_{p-2}),$$

&c. = &c.

$$\alpha^{a^{p-2}} = \frac{1}{p-1}(\beta \cdot c + \beta^2 \cdot c_1 + \ldots + \beta^{p-1} \cdot c_{p-2}).$$

Application of this method to the solution of the equations $x^5-1=0$; $x^7-1=0$; $x^{11}-1=0$; $x^{13}-1=0$.

(*Lagrange*, *Equ. Num.* Note 14; *G. A.* 77; *Leg.* 435–74.)

General Solution of Equations.

(51.) The method of symmetrical functions is inapplicable to the solution of equations of more than *four* dimensions, because the reduced equation is of $1.2.3\ldots(n-2)$ dimensions, if n is a prime number, and of $\dfrac{1.2.3\ldots n}{(p-1)p(1.2.3\ldots q)^p}$ dimensions, if n is composed of two prime factors p, and q; both of which quantities are $> n$, if $n > 4$.

(*Lagrange*, *Equ. Numer.* Note 13.)

Wronski has given what he *asserts* to be a general solution of equations of all degrees.

(*Résolution générale des Equations.*)

Torriani has published a Memoir, the object of which is to shew that Wronski's solution is incorrect.

(*Hist. da Acad. de Lisb. Tom.* 6.)

A general solution of equations of the fifth degree cannot be obtained.

(*Abel*, *Bulletin Univ. des Sciences*; *Ann.* 1826.)

A general solution of equations of any degree superior to the fourth cannot be obtained.

(*Ruffini*, *Theor. delle Equaz.* Cap. xiii.)

Indeterminate Analysis.

EQUATIONS OF THE FIRST DEGREE.

(52.) *Solution of one equation containing two unknown quantities.*

Let the equation be $ax+by=c$. (1)

If a and b have a common factor c, this must also be a factor of e, otherwise the equation is impossible in integers; a will therefore be considered prime to b.

The above equation is always possible if $c > ab - a - b$.

(*B*. 41.)

First Method. Let the value of y obtained from (1) be

$$y = \frac{e - ax}{b} = f - gx + \frac{e_1 - a_1 x}{b},$$

$f - gx$ being the integral part of the fraction $\dfrac{e - ax}{b}$.

Assume $e_1 - a_1 x = bu$,

then $x = \dfrac{e_1 - bu}{a_1} = f_1 - g_1 u + \dfrac{e_2 - a_2 u}{a_1}$;

$$e_2 - a_2 u = a_1 u_1,$$

$$u = \frac{e_2 - a_1 u_1}{a_2} = f_2 - g_2 u_1 + \frac{e_3 - a_3 u_1}{a_2};$$

$$e_3 - a_3 u_1 = a_2 u_2,$$

&c. = &c.

$$e_{n+1} - a_{n+1} u_{n-1} = a_n u_n,$$

$$u_{n-1} = \frac{e_{n+1} - a_n u_n}{a_{n+1}} = f_{n+1} - g_{n+1} u_n,$$

from this last equation we obtain a value of u_{n-1} which is an integer for all integral values of u_n; and thence an integral value of u_{n-2}, and so on, until we obtain the values of x, and y,

$$x = A_1 u_n + B_1,$$

$$y = A_2 u_n + B_2.$$

If a and b have different signs, the number of solutions is infinite; in this case A_1 and A_2 are both positive.

If $a \leftrightharpoons b$, the number of solutions is finite; in this case A_1 and A_2 will have different signs. Suppose A_1 negative, then the number of possible solutions cannot exceed the number of values of $u_n < \frac{B_1}{A_1}$.

(*W.* 367—9; *E.* Pt. II. 1—23; *Bour.* 122—9.)

Second Method. [1] Let the equation be $ax - by = \pm 1$. Convert the fraction $\frac{a}{b}$ into a continued fraction, and let $\frac{\mathrm{A}_m}{\mathrm{B}_m}$ be the converging fraction immediately preceding $\frac{a}{b}$,

$$\text{then (Art. 30.) } a\mathrm{B}_m - b\mathrm{A}_m = \pm 1,$$

$$+ \text{ or } -, \text{ according as } \frac{a}{b} > \text{ or } < \frac{y}{x}.$$

$$\text{If } ax - by = 1, \text{ and } a\mathrm{B}_m - b\mathrm{A}_m = -1,$$

the given equation will be satisfied by assuming

$$x = nb - \mathrm{B}_m, \quad y = na - \mathrm{A}_m.$$

If positive values only of x and y are required, we must have

$$n > \frac{\mathrm{B}_m}{b}, \text{ and } > \frac{\mathrm{A}_m}{a}.$$

If $ax - by = -1$, and $a\mathrm{B}_m - b\mathrm{A}_m = 1$, the values of x, and y may be similarly obtained.

$$\text{If } ax - by \leftrightharpoons a\mathrm{B}_m - b\mathrm{A}_m,$$

it will be sufficient to assume

$$x = nb + \mathrm{B}_m, \quad y = na + \mathrm{A}_m.$$

[2] Let the equation be

$$ax - by = \pm c;$$

and as before let $\frac{\mathrm{A}_m}{\mathrm{B}_m}$ be the converging fraction immediately preceding $\frac{a}{b}$. The general values of x, and y are

$$x = nb \pm c\mathrm{B}_m, \quad y = na \pm c\mathrm{A}_m;$$

$+$ or $-$, according as $a\mathrm{B}_m - b\mathrm{A}_m$, and c, have the same or different signs.

[3] Let the equation be

$$ax + by = c.$$

The values of x and y are

$$x = c\mathrm{B}_m - nb, \quad y = na - c\mathrm{A}_m;$$

n being so assumed, that x and y may both be positive.

The number of solutions equals the integral part of

$$\frac{c\mathrm{B}_m}{b} - \frac{c\mathrm{A}_m}{a},$$

unless $c\mathrm{B}_m \div b$, in which case the number of solutions will be one less than the above number.

(*B*. 159—61; *G. A.* 24; *Leg*. 12—4.)

(53.) *Solution of two equations containing three unknown quantities.*

Let the equations be

$$a_1x + b_1y + c_1z = e_1, \qquad (1)$$

$$a_2x + b_2y + c_2z = e_2. \qquad (2)$$

Eliminating z and reducing the resulting equation to its lowest terms, we obtain

$$ax + by = e. \qquad (3)$$

Let the values of x and y in this equation be

$$x = A_1 u + B_1,$$
$$y = A_2 u + B_2;$$

substituting these values in (1) and reducing, we have

$$fz + gu = h; \qquad (4)$$

obtaining the values of z and u in terms of a new indeterminate, v, and substituting for u its value in those of x, and y, the final result is

$$x = C_1 v + E_1,$$
$$y = C_2 v + E_2,$$
$$z = C_3 v + E_3.$$

The conditions of possibility of the equations (1), (2), may be determined from the relations that must exist between the coefficients of (3) and (4), in order that those equations may be possible, (*Art.* 52.)

If $a_1 = b_1 = c_1 = 1$, and $a_2 > b_2 > c_2$, then e_2 must be $> c_2 e_1$, and $< a_2 e_1$, in order that it may be possible to obtain a positive result. (*B.* 164; *E.* 24—30; *Bour.* 136—9.)

(54.) *Solution of one equation containing three unknown quantities.*

Let the equation be

$$ax + by + cz = e,$$

in which at least two of the coefficients, as a, and b, are prime to each other.

$\frac{A_m}{B_m}$ being, as before, the nearest converging fraction to $\frac{a}{b}$, the values of x and y are

$$x = (e - cz)B_m - nb, \quad y = na - (e - cz)A_m;$$

subject however to the following conditions,

$$z < \frac{e}{c}, \; n > \frac{e - cz}{a} A_m, \; n < \frac{e - cz}{b} B_m.$$

If any one of the coefficients is negative the number of solutions is infinite: suppose therefore that they are all three positive; in which case the number is finite.

To determine the number of solutions: let k; f_1, f_2, &c. f_k, be respectively the integral parts of

$$\frac{e}{c};\ \frac{e-c}{b}\mathrm{B}_m - \frac{e-c}{a}\mathrm{A}_m,\ \frac{e-2c}{b}\mathrm{B}_m - \frac{e-2c}{a}\mathrm{A}_m,\ \&\text{c.}$$

$$\frac{e-kc}{b}\mathrm{B}_m - \frac{e-kc}{a}\mathrm{A}_m,$$

then the total number of solutions $=f_1+f_2+$ &c. $+f_k$.

If k is a high number, the following method of determining the number of solutions will be more convenient: let R_1, R_2, &c. R_b, be the fractional parts of the terms

$$\frac{e-c}{b}\mathrm{B}_m,\ \frac{e-2c}{b}\mathrm{B}_m,\ \&\text{c.}\ \frac{e-kc}{b}\mathrm{B}_m,$$

which fractional parts will recur in periods of b terms; and let g be the integral part of $\frac{k}{b}$, and $S=R_1+R_2+$ &c. $+R_b$.

Similarly, let r_1, r_2, &c. r_a, be the period of the fractional parts of

$$\frac{e-c}{a}\mathrm{A}_m,\ \frac{e-2c}{a}\mathrm{A}_m,\ \&\text{c.}\ \frac{e-kc}{a}\mathrm{A}_m,$$

h the integral part $\frac{k}{a}$, and $s=r_1+r_2+$ &c. $+r_a$; then

$$\text{the total number of solutions} = \left\{\frac{\mathrm{B}_m}{b} - \frac{\mathrm{A}_m}{a}\right\}\left\{ke - \tfrac{1}{2}k(k+1)c.\right\}$$

$$-\left\{gS + R_1 + R_2 + \&\text{c.} + R_{k-gb}\right\}$$

$$+\left\{hs + r_1 + r_2 + \&\text{c.} + r_{k-ha}\right\}.$$

observing that b must be substituted for 0 in the series R_1, R_2, &c.

The period of R_1, R_2, &c. will frequently consist of a repetition of shorter periods, in which case the summation of those quantities will be simplified.

If no two of the quantities are prime to each other, suppose $a = ma_1$, and $b = mb_1$ then

$$a_1 x + b_1 y = \frac{e - oz}{m} = f - gz + \frac{e_1 - c_1 z}{m}.$$

The values of z which render $e_1 - c_1 z \div m$ may be found by the solution of the equation

$$mu + c_1 z = e_1. \qquad (B.\ 162,\ 3;\ Bour.\ 140.)$$

(55.) *Conditions of indeterminateness of equations of the first degree.*

If after elimination the values of the unknown quantities appear under the form $\frac{0}{0}$, the equations are indeterminate; and conversely: this is *always* the case when the constants of one equation are equimultiples of the constants of another.

(*Missery, Ann. de Math. Tome* 1.)

(56.) *Solution of* n *equations containing more than* n *unknown quantities.*

Let the equations be

$$\left.\begin{array}{l} {}^1a_1 x_1 + {}^1a_2 x_2 + \text{\&c.} + {}^1a_r x_r = {}^1a, \\ {}^2a_1 x_1 + {}^2a_2 x_2 + \text{\&c.} + {}^2a_r x_r = {}^2a, \\ \qquad \text{\&c.} \qquad\qquad\qquad \text{\&c.} \\ {}^na_1 x_1 + {}^na_2 x_2 + \text{\&c.} + {}^na_r x_r = {}^na. \end{array}\right\} \quad (A)$$

Assume

$$\left.\begin{array}{l} x_1 = {}^1b + {}^1b_1 u_1 + {}^1b_2 u_2 + {}^1b_3 u_3 + \text{\&c.} \\ x_2 = {}^2b + {}^2b_1 u_1 + {}^2b_2 u_2 + {}^2b_3 u_3 + \text{\&c.} \\ \qquad \text{\&c.} \qquad\qquad\qquad \text{\&c.} \\ x_r = {}^rb + {}^rb_1 u_1 + {}^rb_2 u_2 + {}^rb_3 u_3 + \text{\&c.} \end{array}\right\} \quad (B)$$

$\overset{r}{\mathrm{S}}_m {}^s a_m . x_m = {}^s a, \left\{\begin{smallmatrix} s=1 \\ s=n \end{smallmatrix}\right\}$. (*A*) $\quad x_s = {}^s b + \overset{\infty}{\mathrm{S}}_m {}^s b_m . u_m, \left\{\begin{smallmatrix} s=1 \\ s=r \end{smallmatrix}\right\}$. (*B*)

Substituting these values for x_1, x_2, &c. and putting the coefficients of u_1, u_2, &c. separately $=0$, we obtain

$$\left.\begin{array}{l}
{}^1a_1.{}^1b + {}^1a_2.{}^2b + \text{\&c.} + {}^1a_r.{}^rb = {}^1a, \\
{}^2a_1.{}^1b + {}^2a_2.{}^2b + \text{\&c.} + {}^2a_r.{}^rb = {}^2a, \\
\qquad \text{\&c.} \qquad\qquad \text{\&c.} \\
{}^na_1.{}^1b + {}^na_2.{}^2b + \text{\&c.} + {}^na_r.{}^rb = {}^na:
\end{array}\right\} \qquad (C)$$

$$\left.\begin{array}{l}
{}^1a_1.{}^1b_1 + {}^1a_2.{}^2b_1 + {}^1a_3.{}^3b_1 + \text{\&c.} + {}^1a_r.{}^rb_1 = 0, \\
{}^2a_1.{}^1b_1 + {}^2a_2.{}^2b_1 + {}^2a_3.{}^3b_1 + \text{\&c.} + {}^2a_r.{}^rb_1 = 0, \\
\qquad \text{\&c.} \qquad\qquad\qquad\qquad \text{\&c.} \\
{}^na_1.{}^1b_1 + {}^na_2.{}^2b_1 + {}^na_3.{}^3b_1 + \text{\&c.} + {}^na_r.{}^rb_1 = 0;
\end{array}\right\} \qquad (1)$$

$$\left.\begin{array}{l}
{}^1a_1.{}^1b_2 + {}^1a_2.{}^2b_2 + {}^1a_3.{}^3b_2 + \text{\&c.} + {}^1a_r.{}^rb_2 = 0, \\
{}^2a_1.{}^1b_2 + {}^2a_2.{}^2b_2 + {}^2a_3.{}^3b_2 + \text{\&c.} + {}^2a_r.{}^rb_2 = 0, \\
\qquad \text{\&c.} \qquad\qquad\qquad\qquad \text{\&c.} \\
{}^na_1.{}^1b_2 + {}^na_2.{}^2b_2 + {}^na_3.{}^3b_2 + \text{\&c.} + {}^na_r.{}^rb_2 = 0;
\end{array}\right\} \qquad (2)$$

$$\left.\begin{array}{l}
{}^1a_1.{}^1b_3 + {}^1a_2.{}^2b_3 + {}^1a_3.{}^3b_3 + \text{\&c.} + {}^1a_r.{}^rb_3 = 0, \\
{}^2a_1.{}^1b_3 + {}^2a_2.{}^2b_3 + {}^2a_3.{}^3b_3 + \text{\&c.} + {}^2a_r.{}^rb_3 = 0, \\
\qquad \text{\&c.} \qquad\qquad\qquad\qquad \text{\&c.} \\
{}^na_1.{}^1b_3 + {}^na_2\,{}^2b_3 + {}^na_3\,{}^3b_3 + \text{\&c.} + {}^na_r\,{}^rb_3 = 0.
\end{array}\right\} \qquad (3)$$

$$\text{\&c.} \qquad\qquad \text{\&c.} \qquad\qquad \text{\&c.}$$

The number of indeterminate quantities u_1, u_2, &c. will be

$$\frac{r(r-1)(r-2)\ldots(r-n)}{1\,.\,2\,.\,3\,\ldots(n+1)},$$

of which only $\dfrac{(r-1)(r-2)\ldots(r-n)}{1\,.\,2\,\ldots\,n}$ enter into the value of each unknown quantity.

It appears from the equations (C) that ${}^1b$, ${}^2b$, &c. ${}^rb$, may be any system of values that will satisfy the given equations (A).

$\overset{r}{\mathrm{S}}_m\,{}^sa_m.{}^mb = {}^sa$, $\left\{{s=1 \atop s=n}\right\}$. ($C$) $\overset{r}{\mathrm{S}}_m\,{}^sa_m.{}^mb_t = 0$, $\left\{{s=1 \atop s=n}\right\}$, ($t$); $\left\{{t=1 \atop t=t}\right\}$.

[1] If we have a single equation

$$a_1x_1 + a_2x_2 + \text{\&c.} + a_rx_r = a,$$

we need only consider the first equation of the several systems (1), (2), (3), &c. The equations of the same form which are essentially evanescent are

$$\begin{aligned}
&a_1.a_2 - a_2.a_1 + a_3.0 + a_4.0 + \text{\&c.} + a_r.0 = 0,\\
&a_1.a_3 + a_2.0 - a_3.a_1 + a_4.0 + \text{\&c.} + a_r.0 = 0,\\
&a_1.0 + a_2.a_3 - a_3.a_2 + a_4.0 + \text{\&c.} + a_r.0 = 0,\\
&a_1.a_4 + a_2.0 + a_3.0 - a_4.a_1 + \text{\&c.} + a_r.0 = 0,\\
&a_1.0 + a_2.a_4 + a_3.0 - a_4.a_2 + \text{\&c.} + a_r.0 = 0,\\
&a_1.0 + a_2.0 + a_3.a_4 - a_4.a_3 + \text{\&c.} + a_r.0 = 0,\\
&\qquad\text{\&c.} \qquad\qquad\qquad \text{\&c.}
\end{aligned}$$

We may therefore assume

$$\begin{array}{llllll}
{}^1b_1 = a_2, & {}^2b_1 = -a_1, & {}^3b_1 = 0, & {}^4b_1 = 0, & \text{\&c.} & {}^rb_1 = 0;\\
{}^1b_2 = a_3, & {}^2b_2 = 0, & {}^3b_2 = -a_1, & {}^4b_2 = 0, & \text{\&c.} & {}^rb_2 = 0;\\
{}^1b_3 = 0, & {}^2b_3 = a_3, & {}^3b_3 = -a_2, & {}^4b_3 = 0, & \text{\&c.} & {}^rb_3 = 0;\\
{}^1b_4 = a_4, & {}^2b_4 = 0, & {}^3b_4 = 0, & {}^4b_4 = -a_1, & \text{\&c.} & {}^rb_4 = 0;\\
{}^1b_5 = 0, & {}^2b_5 = a_4, & {}^3b_5 = 0, & {}^4b_5 = -a_2, & \text{\&c.} & {}^rb_5 = 0;\\
{}^1b_6 = 0, & {}^2b_6 = 0, & {}^3b_6 = a_4, & {}^4b_6 = -a_3, & \text{\&c.} & {}^rb_6 = 0.\\
 & \text{\&c.} & & & \text{\&c.} &
\end{array}$$

Particular cases of one equation.

$$a_1x_1 + a_2x_2 = a;$$

$$\begin{aligned}
x_1 &= {}^1b + a_2u_1,\\
x_2 &= {}^2b - a_1u_1.
\end{aligned}$$

$$a_1x_1 + a_2x_2 + a_3x_3 = a;$$

$$\begin{aligned}
x_1 &= {}^1b + a_2u_1 + a_3u_2,\\
x_2 &= {}^2b - a_1u_1 + a_3u_3,\\
x_3 &= {}^3b - a_1u_2 - a_2u_3.
\end{aligned}$$

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = a;$$

$$\begin{aligned}
x_1 &= {}^1b + a_2u_1 + a_3u_2 + a_4u_4,\\
x_2 &= {}^2b - a_1u_1 + a_3u_3 + a_4u_5,\\
x_3 &= {}^3b - a_1u_2 - a_2u_3 + a_4u_6,\\
x_4 &= {}^4b - a_1u_4 - a_2u_5 - a_3u_6.
\end{aligned}$$

[2] If we have two equations,

$$\begin{aligned}
{}^1a_1x_1 + {}^1a_2x_2 + {}^1a_3x_3 + \text{\&c.} + {}^1a_rx_r &= {}^1a,\\
{}^2a_1x_1 + {}^2a_2x_2 + {}^2a_3x_3 + \text{\&c.} + {}^2a_rx_r &= {}^2a;
\end{aligned}$$

it will be necessary to consider the two first equations of the systems (1), (2), (3), &c. The values of the quantities b that render these equations essentially evanescent are

$$\begin{array}{llll}
{}^1b_1 = & ({}^1a_2.{}^2a_3 - {}^2a_2.{}^1a_3), & {}^2b_1 = & -({}^1a_1.{}^2a_3 - {}^2a_1.{}^1a_3),\\
{}^3b_1 = & ({}^1a_1.{}^2a_2 - {}^2a_1.{}^1a_2), & {}^4b_1 = & \qquad 0 \qquad, \text{\&c.}\\[1ex]
{}^1b_2 = & ({}^1a_2.{}^2a_4 - {}^2a_2.{}^1a_4), & {}^2b_2 = & -({}^1a_1.{}^2a_4 - {}^2a_1.{}^1a_4),\\
{}^3b_2 = & \qquad 0 \qquad, & {}^4b_2 = & ({}^1a_1.{}^2a_2 - {}^2a_1.{}^1a_2), \text{\&c.}\\[1ex]
{}^1b_3 = & ({}^1a_3.{}^2a_4 - {}^2a_3.{}^1a_4), & {}^2b_3 = & \qquad 0 \qquad,\\
{}^3b_3 = & -({}^1a_1.{}^2a_4 - {}^2a_1.{}^1a_4), & {}^4b_3 = & ({}^1a_1.{}^2a_3 - {}^2a_1.{}^1a_3), \text{\&c.}\\[1ex]
{}^1b_4 = & \qquad 0 \qquad, & {}^2b_4 = & ({}^1a_3.{}^2a_4 - {}^2a_3.{}^1a_4),\\
{}^3b_4 = & -({}^1a_2.{}^2a_4 - {}^2a_2.{}^1a_4), & {}^4b_4 = & ({}^1a_2.{}^2a_3 - {}^2a_2.{}^1a_3), \text{\&c.}\\
 & \text{\&c.} & & \qquad \text{\&c.}
\end{array}$$

Particular cases of two equations.

$$\begin{aligned}
{}^1a_1x_1 + {}^1a_2x_2 + {}^1a_3x_3 &= {}^1a,\\
{}^2a_1x_1 + {}^2a_2x_2 + {}^2a_3x_3 &= {}^2a;
\end{aligned}$$

$$\begin{aligned}
x_1 &= {}^1b + ({}^1a_2.{}^2a_3 - {}^2a_2.{}^1a_3)\,u_1,\\
x_2 &= {}^2b - ({}^1a_1.{}^2a_3 - {}^2a_1.{}^1a_3)\,u_1,\\
x_3 &= {}^3b + ({}^1a_1.{}^2a_2 - {}^2a_1.{}^1a_2)\,u_1.
\end{aligned}$$

$$
\begin{gathered}
{}^1a_1x_1 + {}^1a_2x_2 + {}^1a_3x_3 + {}^1a_4x_4 = {}^1a, \\
{}^2a_1x_1 + {}^2a_2x_2 + {}^2a_3x_3 + {}^2a_4x_4 = {}^2a;
\end{gathered}
$$

$$
\begin{aligned}
x_1 = {}^1b &+ ({}^1a_2.{}^2a_3 - {}^2a_2.{}^1a_3)\,u_1 + ({}^1a_2.{}^2a_4 - {}^2a_2.{}^1a_4)\,u_2 \\
&+ ({}^1a_3.{}^2a_4 - {}^2a_3.{}^1a_4)\,u_3, \\
x_2 = {}^2b &- ({}^1a_1.{}^2a_3 - {}^2a_1.{}^1a_3)\,u_1 - ({}^1a_1.{}^2a_4 - {}^2a_1.{}^1a_4)\,u_2 \\
&+ ({}^1a_3.{}^2a_4 - {}^2a_3.{}^1a_4)\,u_4, \\
x_3 = {}^3b &+ ({}^1a_1.{}^2a_2 - {}^2a_1.{}^1a_2)\,u_1 - ({}^1a_1.{}^2a_4 - {}^2a_1.{}^1a_4)\,u_3 \\
&- ({}^1a_2.{}^2a_4 - {}^2a_2.{}^1a_4)\,u_4, \\
x_4 = {}^4b &- ({}^1a_1.{}^2a_2 - {}^2a_1.{}^1a_2)\,u_2 - ({}^1a_1.{}^2a_3 - {}^2a_1.{}^1a_3)\,u_3 \\
&- ({}^1a_2.{}^2a_3 - {}^2a_2.{}^1a_3)\,u_4,
\end{aligned}
$$

[3] If we have three equations, the equations of condition that give the values of the quantities b, may be obtained in a similar manner from the first three equations of the systems (1), (2), (3), &c.

The simplest case of three equations is

$$
\begin{gathered}
{}^1a_1x_1 + {}^1a_2x_2 + {}^1a_3x_3 + {}^1a_4x_4 = {}^1a, \\
{}^2a_1x_1 + {}^2a_2x_2 + {}^2a_3x_3 + {}^2a_4x_4 = {}^2a, \\
{}^3a_1x_1 + {}^3a_2x_2 + {}^3a_3x_3 + {}^3a_4x_4 = {}^3a;
\end{gathered}
$$

$$
\begin{aligned}
x_1 = {}^1b &+ \{{}^1a_2({}^2a_3.{}^3a_4 - {}^3a_3.{}^2a_4) - {}^1a_3({}^2a_2.{}^3a_4 - {}^3a_2.{}^2a_4) \\
&+ {}^1a_4({}^2a_2.{}^3a_3 - {}^3a_2.{}^2a_3)\}\,u_1, \\
x_2 = {}^2b &- \{{}^1a_1({}^2a_3.{}^3a_4 - {}^3a_3.{}^2a_4) - {}^1a_3({}^2a_1.{}^3a_4 - {}^3a_1.{}^2a_4) \\
&+ {}^1a_4({}^2a_1.{}^3a_3 - {}^3a_1.{}^2a_3)\}\,u_1, \\
x_3 = {}^3b &+ \{{}^1a_1({}^2a_2.{}^3a_4 - {}^3a_2.{}^2a_4) - {}^1a_2({}^2a_1.{}^3a_4 - {}^3a_1.{}^2a_4) \\
&+ {}^1a_4({}^2a_1.{}^3a_2 - {}^3a_1.{}^2a_2)\}\,u_1, \\
x_4 = {}^4b &- \{{}^1a_1({}^2a_2.{}^3a_3 - {}^3a_2.{}^2a_3) - {}^1a_2({}^2a_1.{}^3a_3 - {}^3a_1.{}^2a_3) \\
&+ {}^1a_3({}^2a_1.{}^3a_2 - {}^3a_1.{}^2a_2)\}\,u_1.
\end{aligned}
$$

(*G. A.* 25; *Bezout, Theor. des Equ. Alg.* 195—223.)

(57.) Every square number is of the same form, with regard to the modulus $2a$, or $2a+1$, or $4a$, as *one* of the squares

$$0^2,\ 1^2,\ 2^2,\ 3^2,\ \&c.\ a^2.$$

Table of the possible Forms of square Numbers for every Modulus from 3 to 20.

$3n$	$3n+1$			$14n$	$14n+1$	$+2$	$+4$	$+7$
$4n$	$4n+1$				$+8$	$+9$	$+11$	
$5n$	$5n\pm1$			$15n$	$15n+1$	$+4$	$+6$	$+9$
$6n$	$6n+1$	$+3$	$+4$		$+10$			
$7n$	$7n+1$	$+2$	$+4$	$16n$	$16n+1$	$+4$	$+9$	
$8n$	$8n+1$	$+4$		$17n$	$17n\pm1$	±2	±4	±8
$9n$	$9n+1$	$+4$	$+7$	$18n$	$18n+1$	$+4$	$+7$	$+10$
$10n$	$10n\pm1$	±4	±5		$+13$	$+16$		
$11n$	$11n+1$	$+3$	$+4$	$19n$	$19n+1$	$+4$	$+5$	$+6$
	$+5$	$+9$		$+7$	$+9$	$+11$	$+16$	$+17$
$12n$	$12n+1$	$+4$	$+9$	$20n$	$20n+1$	$+4$	$+5$	$+9$
$13n$	$13n\pm1$	±3	±4		$+16$			

In investigating the possibility of the equation

$$ax^2 \pm by^2 = z^2,$$

[1] a, and b, may be supposed not to contain a square factor;

[2] x, y, and z, may be considered prime to each other;

[3] if the equation is impossible in integers, it is also impossible in fractions.

If $ma+b$ is a possible form, and $ma+c$ an impossible form, of square numbers to the modulus a, the equation

$$(ma+b)x^2 \pm nay^2 = z^2$$

is always possible; and the equation

$$(ma+c)x^2 \pm nay^2 = z^2$$

is always impossible, if n is prime to a.

Table of the Remainders of Squares to every Modulus, not containing a square Factor, from 2 to 51. (*B. p.* 104.)

Mod. (a)	Remainders. (b)
2	1
3	1
5	1 4
6	1 3 4
7	1 2 4
10	1 4 5 6 9
11	1 3 4 5 9
13	1 3 4 9 10 12
14	1 2 4 7 8 9 11
15	1 4 6 9 10
17	1 2 4 8 9 13 15 16
19	1 4 5 6 7 9 11 16 17
21	1 4 7 9 15 16 18
22	1 3 4 7 9 11 12 14 15 16 20
23	1 2 3 4 6 8 9 12 13 16 18
26	1 3 4 9 10 12 14 16 17 22 23 25
29	1 4 5 6 7 9 13 16 20 22 23 24 25 28
30	1 4 6 9 10 15 16 19 21 24 25
31	1 2 4 5 7 8 9 10 14 16 18 19 20 25 28
33	1 3 4 9 12 15 16 22 25 27 31
34	1 2 4 8 9 13 15 16 17 18 19 21 25 26 30 32 33
35	1 4 9 11 14 15 16 21 25 29 30
37	1 3 4 7 9 10 11 12 16 21 25 26 27 28 30 33 34 36
38	1 4 5 6 7 9 11 16 17 19 20 23 24 25 26 28 30 35 36
39	1 3 4 9 10 12 13 16 22 25 27 30 36
41	1 2 4 5 8 9 10 16 18 20 21 23 25 31 32 33 36 37 39 40
42	1 4 7 9 15 16 18 21 22 25 28 30 36 37 39
43	1 4 6 9 10 11 13 14 15 16 17 21 23 24 25 31 35 36 38 40 41
46	1 2 3 4 6 8 9 12 13 16 18 23 24 25 26 27 29 31 32 35 36 39 41
47	1 2 3 4 6 7 8 9 12 14 16 17 18 21 24 25 27 28 32 34 36 37 42
51	1 4 9 13 15 16 18 19 21 23 25 30 34 36 42 43 49

Let a be a prime number, and b prime to a; then if

$$b^2,\ (2b)^2,\ (3b)^2,\ \&c.\ \left(\frac{a-1}{2}b\right)^2,$$

be severally divided by a, the remainders will all be unequal.

The product of a possible and impossible form of squares to the same modulus is always impossible. (*B*. 42—52.)

Conditions of the possibility of the equation

$$x^2 - ay^2 = bz^2,$$

in which a and b are positive integers, and $a < b$.

Let a number c be found such that $c > \frac{1}{2}b$, and $c^2 - a \div b$;

and let the following system of quantities be constructed:

$$c^2 - a = b\ b_1 z_1^2,\ \ c_1 = m_1 b_1 \pm c \ngtr \tfrac{1}{2} b_1;$$

$$c_1^2 - a = b_1 b_2 z_2^2,\ \ c_2 = m_2 b_2 \pm c_1 \ngtr \tfrac{1}{2} b_2;$$

$$c_2^2 - a = b_2 b_3 z_3^2,\ \&c. = \ \ \&c.$$

then if a, b, b_1, are such that *any* integral values of c_1, c_2, e_1, e_2, will satisfy the conditions

$$c_1^2 - a \div b, \qquad e_1^2 - b \div a;$$

$$c_2^2 - a \div b_1, \qquad e_2^2 - b_1 \div a;$$

the equation is always possible.

The equation $ax^2 + by^2 = cz^2$,

in which a, b, and c are prime to each other, is possible, if the conditions $ae_1^2 + b \div c$, $ce_2^2 - b \div a$, $ce_3^2 - a \div b$,

may be fulfilled by *any* integral values of e_1, e_2, e_3.

Hence may be derived the following rule: divide b and c by a, then if both or neither of the remainders are found amongst the remainders of squares to the modulus a, the equation *may* be possible. If this condition is fulfilled, and the same relation exists between the remainders of a, and c, when divided by b, and those of a, and $c-b$, when divided by c, the equation is possible; if either of these three conditions fails, the equation is impossible. (*B.* 53, 178; *Leg.* 23—7.)

Impossible pairs of quadratic equations:

$$\begin{cases} x^2+y^2=z^2, \\ x^2-y^2=u^2. \end{cases} \qquad \begin{cases} x^2+2y^2=2z^2, \\ x^2-2y^2=2u_2. \end{cases}$$

$$\begin{cases} x^2+y^2=2z^2, \\ x^2-y^2=2u^2. \end{cases} \qquad \begin{cases} x^2+2y^2=z^2, \\ x^2-2y^2=u^2. \end{cases}$$

$$\begin{cases} 2x^2+y^2=z^2, \\ 2x^2-y^2=u^2. \end{cases} \qquad \begin{cases} 2x^2+y^2=2z^2, \\ 2x^2-y^2=2u^2. \end{cases}$$

The pair of equations $\begin{cases} x^2 \pm cy^2 = z^2, \\ x^2 \pm \ \ y^2 = u^2, \end{cases}$

is impossible, if the two equations

$$m^2 \pm cn^2 = (c-1)p^2,$$

$$m^2 \pm \ \ n^2 = (c-1)p^2,$$

are separately impossible. (*B.* 55, 6.)

INDETERMINATE ANALYSIS.

EQUATIONS OF THE SECOND DEGREE.

(58.) *Solution of particular equations.*

Let the given equation be $x^2+1=y^2$.

First Method: assume $y=x+\frac{m}{n}$, n being $>m$, then

$$x=\frac{n^2-m^2}{2mn}.$$

Second Method: assume $y = 1 + \frac{mx}{n}$, then

$$x = \frac{2mn}{m^2 - n^2}.$$

Two square numbers whose sum, or whose difference, is also a square may be found from the equation

$$(2mn)^2 + (n^2 - m^2)^2 = (n^2 + m^2)^2.$$

To find the values of x which satisfy the equation

$$a + bx + cx^2 = y^2.$$

[1] Suppose $c \text{ ♃ } f^2$; assume $(a + bx + f^2x^2)^{\frac{1}{2}} = fx + \frac{m}{n}$, then

$$x = \frac{m^2 - n^2a}{n^2b - 2mnf}.$$

[2] Suppose $a \text{ ♃ } f^2$; assume $(f^2 + bx + cx^2)^{\frac{1}{2}} = f + \frac{mx}{n}$, then

$$x = \frac{2mnf - n^2b}{n^2c - m^2}.$$

[3] Suppose $a = 0$; assume $(bx + cx^2)^{\frac{1}{2}} = \frac{mx}{n}$, then

$$x = \frac{bn^2}{m^2 - cn^2}.$$

[4] Suppose $a + bx + cx^2 = (f + gx).(h + kx)$;

assume $\{(f + gx)(h + kx)\}^{\frac{1}{2}} = \frac{m}{n}(f + gx)$, then

$$x = \frac{fm^2 - hn^2}{kn^2 - gm^2}.$$

In this case $b^2 - 4ac$ must be a square.

(*E.* 38—62; *B.* 167—70.)

(59.) *Reduction of the general equation.*

The most general form of a quadratic equation is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

which by assuming $(by + d)^2 - 4a(cy^2 + ey + f) = t,$

$$b^2 - 4ac = A, \quad bd - 2ae = g, \quad d^2 - 4af = h;$$

$$\text{and } Ay + g = u, \quad g^2 - Ah = B;$$

may always be reduced to the form

$$u^2 - At^2 = B. \qquad (1)$$

from the solution of which the values of x and y may be obtained; for $y = \frac{u-g}{A}$,

$$x = \frac{(t-d)A - (u-g)b}{2Aa}.$$

In these values of x and y, t and u may be either positive or negative, integral or fractional; in its most general form, the equation (1), by substituting $\frac{x}{z}$ and $\frac{y}{z}$, for u and t, becomes

$$x^2 - Ay^2 = Bz^2.$$

In all cases in which this equation is possible, it may be transformed into another of the form

$$x_1^2 - y_1^2 = cz_1^2.$$

(*G. A.* 26, 7; *B.* 172, 3; *Leg.* 15, 16.)

(60.) *Reduction of the equation* $x^2 - ay^2 = bz^2$.

The following conditions may be supposed to exist:

[1] x, y, and z, are prime to each other.

[2] y is prime to b, and z to a.

[3] a, and b, have no quadratic factor.

[4] a, and b, are both positive.

[5] $b > a$.

Assume $$x = ny - by_1,$$
n being such that $n^2 - a \div b$: let the quotient be $b_1 k^2$, where b_1 contains no quadratic factor; then
$$b_1 k^2 y^2 - 2nyy_1 + by_1^2 = z^2.$$
Multiply this equation by $b_1 k^2$, and assume
$$b_1 k^2 y - ny_1 = x_1, \qquad \text{and } k^2 z^2 = z_1^2,$$
then $$x_1^2 - ay_1^2 = b_1 z_1^2,$$

which is similar to the original equation, except that $b_1 < \frac{1}{4} b$. If b_1 is equal to unity, or to any square, the required transformation is effected: the values of x, y, and z, are

$$x = \frac{nx_1 + n^2 y_1}{b_1 k^2} - by_1,$$

$$y = \frac{x_1 + ny_1}{b_1 k^2}, \qquad (1)$$

$$z = \frac{z_1}{k}.$$

If b_1 is not a square, and $> a$, the same process may be repeated, and we obtain
$$x_2^2 - ay_2^2 = b_2 z_2^2.$$
In which equation $b_2 < \frac{1}{4} b_1$: by pursuing this method we must at length arrive at an equation in which b_m is either 1, a square, or $\nless a$. In the latter case, by transposing, putting c for b_m, and neglecting the subscript indices of x, y, and z, we obtain $$x^2 - cz^2 = ay^2.$$

Proceeding as before, this may be reduced to another equation $$x^2 - cz^2 = a_m y^2,$$
in which $a_m < c$. Then similarly putting e for a_m, we have
$$x^2 - ey^2 = cz^2.$$

By this process, the coefficient of either y or z must be reduced to a square, or to unity: the indeterminate quantities in the equation last obtained may be expressed in terms of x, y, and z, by successive substitution by means of the equations (1). The equation is thus reduced to

$$x^2 - y^2 = az^2.$$

The only systems of quantities to be computed are

$$\begin{array}{c|c}
\dfrac{n^2-a}{b} = b_1k^2 & x^2 - ay^2 = b_1z^2, \\
\dfrac{n_1^{\,2}-a}{b_1} = b_2k_1^{\,2} & x^2 - ay^2 = b_2z^2, \\
\text{\&c.} = \text{\&c.} & \text{\&c.} = \text{\&c.} \\
\dfrac{n_{m-1}^{\;\;2}-a}{b_{m-1}} = b_m k_{m-1}^{\;\;2} & x^2 - ay^2 = b_m z^2 = cz^2.
\end{array}$$

$$\begin{array}{c|c}
\dfrac{n^2-c}{a} = a_1l^2 & x^2 - cz^2 = a_1y^2, \\
\dfrac{n_1^{\,2}-c}{a_1} = a^2l_1^{\,2} & x^2 - cz^2 = a_2y^2, \\
\text{\&c.} = \text{\&c.} & \text{\&c.} = \text{\&c.} \\
\dfrac{n_{m-1}^{\;\;2}-c}{a_{m-1}} = a_m l_{m-1}^{\;\;2} & x^2 - cz^2 = a_m y^2 = ey^2. \\
\text{\&c.} & \text{\&c.}
\end{array}$$

in which k^2, l^2, &c. are the greatest integral squares in the quotients $\frac{n^2-a}{b}$, $\frac{n^2-c}{a}$, &c.

(*Lagr. Mem. Berl.* 1767; *B.* 174–6; *Leg.* 15–22; *E. Add.* 52.)

(61.) Solution of the equation $x^2 - y^2 = az^2$.

Let $a = a_1 . a_2$, and $z = z_1 . z_2$, then

$$x = \tfrac{1}{2}(a_1 . z_1^{\,2} + a_2 . z_2^{\,2}),$$
$$y = \tfrac{1}{2}(a_1 . z_1^{\,2} - a_2 . z_2^{\,2}). \qquad (B.\ 54;\ Leg.\ 17.)$$

(62.) Solution of the equation $x^2 - ay^2 = \pm 1$.

Let $\sqrt{a}$ be expressed by a continued fraction, (Art. 31.) and let $\frac{A_m}{B_m}$ be any converging fraction corresponding to the quotient $2e$; the above equation will be satisfied by the values

$$x = A_m, \qquad y = B_m.$$

If the period consists of an even number of quotients, the equation $x^2 - ay^2 = 1$ will be satisfied by every converging fraction corresponding to a complete period, and $x^2 - ay^2 = -1$ will be impossible.

If the period consists of an odd number of quotients, the equations $x^2 - ay^2 = -1$, and $x^2 - ay^2 = 1$, will be alternately satisfied by the converging fractions corresponding to complete periods.

(*B.* 150; *E.* 96—111, *Add.* 37.)

Having given one solution of the equation

$$x^2 - ay^2 = \pm 1,$$

to determine the general values of x and y.

[1] $x^2 - ay^2 = 1$, and $p^2 - aq^2 = 1$;

$$x = \tfrac{1}{2}\{\overline{p+q\sqrt{a}}|^{m} + \overline{p-q\sqrt{a}}|^{m}\},$$

$$y = \frac{1}{2\sqrt{a}}\{\overline{p+q\sqrt{a}}|^{m} - \overline{p-q\sqrt{a}}|^{m}\}.$$

[2] $x^2 - ay^2 = 1$, and $p^2 - aq^2 = -1$;

$$x = \tfrac{1}{2}\{\overline{p+q\sqrt{a}}|^{2m} + \overline{p-q\sqrt{a}}|^{2m}\},$$

$$y = \frac{1}{2\sqrt{a}}\{\overline{p+q\sqrt{a}}|^{2m} - \overline{p-q\sqrt{a}}|^{2m}\}.$$

[3] $x^2 - ay^2 = -1$, and $p^2 - aq^2 = -1$;

$$x = \tfrac{1}{2}\{\overline{p+q\sqrt{a}}|^{2m-1} + \overline{p-q\sqrt{a}}|^{2m-1}\},$$

$$y = \frac{1}{2\sqrt{a}}\{\overline{p+q\sqrt{a}}|^{2m-1} - \overline{p-q\sqrt{a}}|^{2m-1}\}. \quad (B. 180.)$$

Table of the least Values of x and y in the Equation $x^2 - ay^2 = 1$, for all Values of a from 2, to 99. (*E. v.* 2; *p.* 89.)

a	x	y	a	x	y
2	3	2	53	66249	9100
3	2	1	54	485	66
5	9	4	55	89	12
6	5	2	56	15	2
7	8	3	57	151	20
8	3	1	58	19603	2574
10	19	6	59	530	69
11	10	3	60	31	4
12	7	2	61	1766319049	226153980
13	649	180	62	63	8
14	15	4	63	8	1
15	4	1	65	129	16
17	33	8	66	65	8
18	17	4	67	48842	5967
19	170	39	68	33	4
20	9	2	69	7775	936
21	55	12	70	251	30
22	197	42	71	3480	413
23	24	5	72	17	2
24	5	1	73	2281249	267000
26	51	10	74	3699	430
27	26	5	75	26	3
28	127	24	76	57799	6630
29	9801	1820	77	351	40
30	11	2	78	53	6
31	1520	273	79	80	9
32	17	3	80	9	1
33	23	4	82	163	18
34	35	6	83	82	9
35	6	1	84	55	6
37	73	12	85	285769	30996
38	37	6	86	10405	1122
39	25	4	87	28	3
40	19	3	88	197	21
41	2049	320	89	500001	53000
42	13	2	90	19	2
43	3482	531	91	1574	165
44	199	30	92	1151	120
45	161	24	93	12151	1260
46	24335	3588	94	2143295	221064
47	48	7	95	39	4
48	7	1	96	49	5
50	99	14	97	62809633	6377352
51	50	7	98	99	10
52	649	90	99	10	1

(63.) Solution of the equation $x^2 - ay^2 = \pm b$,
in which $b < \sqrt{a}$.

If this equation is possible, b will be found amongst the denominators of the complete quotients (a_1, a_2, a_3, &c. Art. 31.) of the converging fractions which express the value of $\sqrt{a}$.

Given one solution of the above equation to determine the general values of x and y.

Suppose that we have found, m, n, p, q, such that

$$m^2 - an^2 = \pm b, \text{ and } p^2 - aq^2 = \pm 1,$$

then
$$x = mp \pm anq,$$
$$y = np \pm mq,$$

in which the general values of x and y obtained in the preceding Art. must be substituted for p and q respectively.

The equation $p^2 - aq^2 = 1$, or $p^2 - aq^2 = -1$ must be employed, according as the known solution and the given equation have the same or contrary signs.

To determine the general values of x and y in the above equation, b being $> \sqrt{a}$.

[1] Suppose b to be composed of factors b_1, b_2, &c. each $< \sqrt{a}$: the solution of the given equation, when possible, may be deduced from those of the equations

$$m_1{}^2 - an_1{}^2 = \pm b_1, \quad m_2{}^2 - an_2{}^2 = \pm b_2, \text{ \&c.}$$

for $(m_1{}^2 - an_1{}^2)(m_2{}^2 - an_2{}^2)(m_3{}^2 + an_3{}^2)$ &c. $= x^2 - ay^2$.

Having found one integral value of x and y, the general values may be found as before by means of the equation

$$p^2 - aq^2 = \pm 1.$$

[2] Suppose b not to be composed of factors $< \sqrt{a}$: it will in this case be necessary to find the values of t and u in the equation

$$t^2 - au^2 = \pm bz^2,$$

then putting $m = \frac{t}{x}$, $n = \frac{u}{x}$ the equation becomes

$$m^2 - an^2 = \pm b;$$

calling this the known solution, the general values may be found as before. In this case however the values may be fractional.

(*B*. 181—4.)

(64.) Solution of the equation $ax^2 + bxy + cy^2 = \pm e$.

In this equation, x and y, as well as y and e, may be considered prime to each other.

This may be reduced to a similar equation of a simpler form, by assuming

$$\begin{aligned} x &= a_1 x_1 + b_1 y_1, \\ y &= a_2 x_1 + b_2 y_1; \\ A &= a a_1^2 + b a_1 a_2 + c a_2^2, \\ B &= 2(a a_1 b_1 + c a_2 b_2) + b(a_1 b_2 + a_2 b_1), \\ C &= a b_1^2 + b b_1 b_2 + c b_2^2. \end{aligned}$$

a_1, a_2, and b_1, b_2, being so assumed that $a_1 b_2 - a_2 b_1 = \pm 1$.

If $b > a$, and $> c$, the equation may be transformed into another in which $b \ngtr a$, or c.

Suppose $a < c$; assume $x = x_1 - my$,

m being the nearest integer to $\frac{b}{a}$; assume also

$$b_1 = am - b, \text{ and } c_1 = am^2 - bm + c,$$

the equation becomes $a x_1^2 - b_1 x_1 y + c_1 y^2 = \pm e$, in which $b_1 \ngtr a$.

If $b_1 > c_1$ the same process must be repeated.

In the successive transformed equations, the value of $b^2 - 4ac$ is always the same.

If $b^2 - 4ac < 0$, a and c are the least numbers contained in the transformed equation, in which $b \ngtr a$, or c.

(*Leg*. 53—8; *B*. 100—2.)

[1] Suppose $b^2 - 4ac < 0$, and $= -f$.

Multiplying the equation by $4a$, and assuming

$$z = 2ax + by,$$

we obtain $$z^2 + fy^2 = 4ae.$$

If any integral value of $y < 2\left(\frac{ae}{f}\right)^{\frac{1}{2}}$ which satisfies this equation, also renders $by \pm z \div 2a$, a solution may be obtained; if both these conditions are not fulfilled, the equation is impossible in integers.

[2] Suppose $b^2 - 4ac = 0$.

In this case the first first side of the equation becomes a perfect square, and is therefore possible, if e is a square.

[3] Suppose $b^2 - 4ac > 0$, and $= k^2$.

In this case the proposed equation may be decomposed into two simple equations

$$a_1 x + b_1 y = e_1, \quad \text{and } a_2 x + b_2 y = e_2.$$

[4] Suppose $b^2 - 4ac = 4f$, f not being a square.

First; let e be $< \sqrt{f}$: expand a root of the equation

$$ax^2 + bx + c,$$

in a continued fraction. (*p.* 32.) If amongst the quantities $2a_1$, $2a_2$, &c. the denominators of the complete quotients, we find the number e, the numerator and denominator of the corresponding converging fraction, if substituted for x and y respectively, will satisfy the given equation.

As often as e occurs, a different solution may be obtained; if it does not occur at all, there is no integral solution of the proposed equation.

Secondly; suppose $e > \sqrt{f}$: assume

$$y = nx_1 + ey_1,$$

$$a_1 = \frac{an^2 + bn + c}{e},$$

$$b_1 = 2na + b,$$

$$c_1 = ae;$$

the equation is reduced to $a_1 x_1^2 + b_1 x_1 y_1 + c_1 y_1^2 = \pm 1$, which must be solved for each value of n.

The given equation is impossible in integers, unless n may be so assumed, that a_1 may be an integer. (*Leg.* 75—83.)

Solution of the equation $ax^2 + bxy + cy^2 = \pm 1$.

Let it be transformed so as to fulfil the conditions

$$b \ngtr a, \text{ or } c, \text{ and } a < c;$$

multiply the equation by a, and assume $z = ax + by$, the equation is reduced to $z^2 + ey^2 = \pm a$,

which may be solved by preceding methods. (*E. Add.* 66—71.)

FORMS OF CUBES.

(65.) Every cube $\mathrel{⋕} 4n$, or $4n \pm 1$;

$\mathrel{⋕} 7n$, or $7n \pm 1$;

$\mathrel{⋕} 9n$, or $9n \pm 1$.

All cubes are of the same form to modulus a, as the cubes

$$0^3,\ 1^3,\ 2^3, \text{ \&c. } (a-1)^3.$$

$a^3 \mathrel{⋕} a$ to modulus 6.

The equations $x^3 \pm y^3 = z^3$,

$$\frac{x^3}{a^3} \pm \frac{y^3}{b^3} = \frac{z^3}{c^3},$$

$$x^3 \pm y^3 = 2z^3,$$

and generally $mx^3 \pm nay^3 = z^3$,

in which m is a number of an impossible form to modulus a, and n prime to a, are impossible in integers.

No triangular number < 1 is a cube.

(*B.* 60—70; *Leg.* 328—32.)

(66.) The equation $a+a_1x+a_2x^2+a_3x^3=y^2$ admits of a direct solution only when $a=b^2$.

First Method. Assume

$$b^2+a_1x+a_2x^2+a_3x^3=(b+cx)^2,$$

and $$a_1=2bc;$$

then $$x=\frac{c^2-a_2}{a_3}=\frac{a_1{}^2-4b^2a_2}{4b^2a_3}.$$

Second Method. Assume

$$b^2+a_1x+a_2x^2+a_3x^3=(b+cx+ex^2)^2,$$

and $$a_1=2bc, \quad a_2=c^2+2be;$$

then $$x=\frac{a_3-2ce}{e^2}.$$ (*E.* 26, 7.)

If a is not a square, one solution must be obtained by trial, let this be $a+a_1h+a_2h^2+a_3h^3=k^2$, then by assuming $x=y+h$, the equation is reducible to the above form. (*B.* 184, 5; *E. Pt.* II, 112—27.)

Solution of the equation $a+a_1x+a_2x^2+a_3x^3=y^3$.

[1] Suppose $a=b^3$:

Assume $$b^3+a_1x+a_2x^2+a_3x^3=(b+ex)^3,$$

then $$e=\frac{a_1}{3b^2}, \quad \text{and } x=\frac{a_2-3be^2}{e^3-a_3}.$$

[2] Suppose $a_3=c^3$:

assume $$a+a_1x+a_2x^2+c^3x^3=(e+cx)^3,$$

then $$e=\frac{a_2}{3c^2}, \quad \text{and } x=\frac{a-e^3}{3ce^2-a_1}.$$

[3] Suppose $a=b^3$, and $a_3=c^3$:

assume $$b^3+a_1x+a_2x^2+c^3x^3=(b+cx)^3,$$

then $$x=\frac{a_1-3b^2c}{3bc^2-a_2}.$$

If neither of these conditions is fulfilled, one solution must be obtained; let this be

$$a + a_1 h + a_2 h^2 + a_3 h^3 = k^3,$$

then by assuming $x = h + y$, the equation may be reduced to the form [1]. (*E.* 147—61; *B.* 191—5.)

Solution of the equation $ax^2 + cy^2 = z^3$.

Assume $x = ap^3 - 3cpq^2$, and $y = 3ap^2q - cq^3$, then

$$z = ap^2 + cq^2. \qquad (E.\ 187.)$$

Solution of the equation $x^2 + axy + by^2 = z^3$.

Assume $x = t^3 - 3btu^2 - abu^3$,

$y = 3t^2u + 3atu^2 + (a^2 - b)u^2$;

then $z = t^2 + atu + bu^2$. (*B.* 196.)

Forms of Biquadrates.

(67.) Every even biquadrate $\pitchfork 2^4.n$.

Every odd biquadrate $\pitchfork 2^4.n + 1$.

All 4th powers are of the same form to modulus a as

$0^4, 1^4, 2^4$, &c. $(\tfrac{1}{2}a)^4$, when a is even,

$0^4, 1^4, 2^4, \overline{\tfrac{1}{2}(a-1)}\big|^4$, when a is odd.

Remainders of 4th powers from every modulus from 3 to 12.

Mod.	Rem.	Mod.	Remainders.	Mod.	Remainders.
3	1	6	1 3 4	9	1 4 7
4	1	7	1 2 4	10	1 5 6
5	1	8	1	11	1 3 4 5 9
				12	1 4 9

Impossible forms of equations of the fourth degree.

$$x^4 \pm y^4 = z^4,$$

$$x^4 \pm a^2 y^4 = z^4,$$

and generally, $mx^4 \pm nay^4 = z^4$,
m being a number of an impossible form to modulus a, and n prime to a.

No triangular number > 1 is a biquadrate.

(*Leg.* 324—7; *B.* 71—6.)

INDETERMINATE EQUATIONS OF THE FOURTH DEGREE.

(68.) Solution of the equation $a + a_1x + a_2x^2 + a_3x^3 + a_4x^4 = y^2$.

[1] Suppose $a = b^2$:
assume $b^2 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 = (b + ex + fx^2)^2$,

and $$e = \frac{a_1}{2b}, \quad f = \frac{a_2 - e^2}{2b};$$

then $$x = \frac{a_3 - 2ef}{f^2 - a_4}.$$

[2] Suppose $a_4 = c^2$:
assume $a + a_1x + a_2x^2 + a_3x^3 + c^2x^4 = (f + ex + cx^2)^2$,

and $$e = \frac{a_3}{2c}, \quad f = \frac{a_2 - e^2}{2c};$$

then $$x = \frac{f^2 - a}{a_1 - 2ef}.$$

[3] Suppose $a = b^2$, and $a_4 = c^2$:
assume $b^2 + a_1x + a_2x^2 + a_3x^3 + c^2x^4 = (b + ex + cx^2)^2$,

then $$e = \frac{a_1}{2b}, \text{ and } x = \frac{e^2 \pm 2bc - a_2}{a_3 \mp 2ce};$$

or $$e = \frac{a_3}{2c}, \text{ and } x = \frac{a_1 \pm 2be}{e^2 \mp 2bc - a_2}.$$

If neither of these conditions exists, let *one* solution be found,

$$a + a_1h + a_2h^2 + a_3h^3 + a_4h^4 = k^2,$$

from which, and the given equation,

$$x = \frac{h}{4a^2 - 3k^4}(k^4 - 8ak^2 + 4a^2).$$

If the equation is $a + a_2x^2 + a_4x^4 = y^2$,

let one solution be found,

$$a + a_2h^2 + a_4h^4 = k^2;$$

then if $p = \dfrac{a_2h + 2a_4h^3}{k}$, and $q = \dfrac{a_2 + 6a_4h^2 - p^2}{2k}$,

$$x = \frac{(q^2 + 3a_4)h - 2pq}{q^2 - a_4}. \qquad (E.\ 128\text{–}46;\ B.\ 186\text{–}90.)$$

Solution of the equation $x^2 + cy^2 = z^4$.

Assume $x = p^4 - 6cp^2q^2 + c^2q^4$,

$y = 4pq(p^2 - cq^2)$;

then $z = p^2 + cq^2$. (*E*. 198.)

Solution of the equation $x^2 + axy + by^2 = z^4$.

Assume $x = t^4 - 6bt^2u^2 - 4abtu^3 - (a^2 - b)bu^4$,

$y = 4t^3u + 6at^2u^2 + 4(a^2 - b)tu^3 + (a^2 - 2b)au^4$;

then $z = t^2 + atu + bu^2$. (*B*. 197.)

(69.) *Solution of the homogeneous equation*

$$ax^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \&c. + a_ny^n = \pm e,$$

or

$$f(x, y) = \pm e;$$

in which x and y, as well as y and b, may be considered prime to each other.

Assume $x = cy + ez$, c being such that

$$ac^n + a_1c^{n-1} + a_2c^{n-2} + \&c. + a_n \subset\!\div\!\supset e;$$

by substituting this value of x, the given equation is reduced to

$$by^n + b_1y^{n-1}z + b_2y^{n-2}z^2 + \&c. + b_nz^n = \pm 1.$$

There cannot be more than n values of x, between the limits $+\frac{1}{2}e$ and $-\frac{1}{2}e$, that render the integral polynomial

$$\phi(x) \subset\!\div\!\supset e. \qquad (B.\ 88.)$$

To determine p and q, the values of y and z, which render

$f(y, z)$ a *minimum:*

let a_1, a_2, &c. $a_r + \beta_1\sqrt{-1}$, &c. be the roots of the equation

$$bx^n + b_1x^{n-1} + b_2x^{n-2} + \text{\&c.} + b_n = 0,$$

then $\frac{p}{q}$ is the converging fraction nearest to *one* of the quantities a_1, a_2, &c. a_r, &c. which must be determined by experiment.

(*Leg.* 120—8. *E. Add.* 28.)

(70.) *Solution of the equation* $x^n - b = ay$, a being a prime number, and b prime to a.

If n and $a-1$ have a common factor c, there will be c solutions, or none.

If the given equation is possible, $b^{\frac{a-1}{c}} - 1 \div a$.

If one solution, $x = e$, has been obtained, the others may be found by multiplying e by the several roots of the equation

$$x^n - 1 = ay.$$

The proposed equation is always possible, if n is prime to $a-1$.

Solution of the equation $x^n - 1 = ay$.

If n is prime to $a-1$, the only possible solution is $x = 1$.

Let $a - 1 = en$, then $x = z^e$, z being any number prime to a.

If u is the remainder of $z^e \div a$, all the values of x will be found amongst the remainders of the quantities

$$u,\ u^2,\ u^3,\ \text{\&c.}\ u^{n-1} \div a,$$

unless two or more of these remainders are equal.

If $r^m - 1 = ay$ is impossible, r being a root of the equation $x^n - 1 = ay$, and m a divisor of n, then r is a *primitive* root.

If n_1, n_2, &c. are the prime factors of n, the number of primitive roots of the equation $x^n - 1 = ay$ will be expressed by

$$n \cdot \frac{n_1 - 1}{n_1} \cdot \frac{n_2 - 1}{n_2}.\ \text{\&c.}$$

If n is a prime number, every number prime to a is a primitive root.

If $n = n_1^{p_1}.n_2^{p_2}$. &c. and m_1, m_2, &c. are the remainders of $n_1^{p_1}$, $n_2^{p_2}$, &c. $\div\, a$, and r_1, r_2, &c. the roots of the equations

$$x^{m_1} - 1 = ay, \qquad x^{m_2} - 1 = ay, \text{ \&c.}$$

the values of x are the powers of r_1, r_2, &c.

Solution of the equation $x^{2n} + 1 = ay$, a being a prime number, and $a - 1 \div 4n$.

Let r^m be the general root of the equation

$$x^{4n} - 1 = ay,$$

then r^{2p+1} will be the general value of x.

(*Leg.* 333—41; *B.* 200—4.)

Solution of the equation $x^n - b = ay$,

[1] when $b^m \pm 1 \div a$, m being a divisor of $\dfrac{a-1}{n}$;

[2] when $b^m - 1 \div a$. (*Leg.* 342—9; *B.* 205, 6.)

Solution of the equation $x^2 + a = 2^n y$. (*Leg.* 350—2.)

THEORY OF NUMBERS.

(71.) Properties of the divisors of numbers. (*E.* 58–67; *B.* 1–15.)

Any number a may be represented by $a_1^{r_1}.a_2^{r_2}.a_3^{r_3}\ldots a_n^{r_n}$; a_1, a_2, &c. a_n, being prime numbers.

The number of the divisors of $a = (r_1+1)(r_2+1)\ldots(r_n+1)$.

The sum of all the divisors

$$= \frac{a_1^{r_1+1} - 1}{a_1 - 1} . \frac{a_2^{r_2+1} - 1}{a_2 - 1} \ldots \frac{a_n^{r_n+1} - 1}{a_n - 1}.$$

The number of integers $< a$, and prime to that number

$$= a \frac{a_1 - 1}{a_1} . \frac{a_2 - 1}{a_2} \ldots \frac{a_n - 1}{a_n}.$$

Any number prime to a and b, is prime to ab.

(*B.* 21; *Leg.* vi—xiv.)

(72.) Nature of prime numbers. (*E.* 37—44.)

All prime numbers $>2 \pitchfork 4n \pm 1$; and $>3 \pitchfork 6n \pm 1$.

No algebraical formula can contain prime numbers only.

The number of prime numbers is infinite.

If a is any number whatever, and b_1, b_2, b_3, &c. the numbers prime to, and $<2a$, all prime numbers will be of one or other of the forms

$$4an \pm b_1, \quad 4an \pm b_2, \quad 4an \pm b_3, \quad \&c.$$

The number of times that any prime factor a occurs in the product $1.2.3\ldots n$ is the sum of the integral parts of the fractions $\frac{n}{a}, \frac{n}{a^2}, \frac{n}{a^3}$, &c.

There cannot be n prime numbers in arithmetical progression, unless their common difference is divisible by the product of all prime numbers up to n; except when n is the first term of the series, and then there cannot be more than n.

The sum of any number of prime numbers is a composite number.

If a and b are prime to each other, and each term of the series $b, 2b, 3b$, &c. $(a-1)b$, be divided by a, the remainders will all be unequal.

(*B.* 29—39; *Leg.* xvi—xxiii.)

If n is a prime number, $1.2.3\ldots(n-1)+1 \div n$,

and $(x+1)(x+2)(x+3)\ldots(x+n-1)-x^{n-1}+1 \div n$.

If x is prime to n,

$x^n \pitchfork x$, to modulus n;

$x^{n-1}-1 \div n$;

$x^{n-1} \pitchfork an$, or $an \pm 1$;

$x^{\frac{1}{2}(n-1)} \pitchfork an$, or $an \pm 1$.

(*Lagrange*, *Mem. de l'Acad. Berlin*, *Ann.* 1771; *B.* 86, 7.)

If 2^n-1 is a prime number, $2^{n-1}(2^n-1)$ is a perfect number.

The only *known* perfect numbers are

$2\,(2^2-1)=6,$	$2^{12}(2^{13}-1)=33550336,$
$2^2(2^3-1)=28,$	$2^{16}(2^{17}-1)=8589869056,$
$2^4(2^5-1)=496,$	$2^{18}(2^{19}-1)=137438691328,$
$2^6(2^7-1)=8128,$	$2^{30}(2^{31}-1)=2305843008139952128.$

If 3.2^n-1, 6.2^n-1, and $18.2^{2n}-1$, are prime numbers,

$$2^{n+1}(18.2^{2n}-1),\text{ and }2^{n+1}(3.2^n-1)(6.2^n-1),$$

are amicable numbers.

The only *known* amicable numbers are

$(n=1)$	220	and 284,
$(n=3)$	17296	 18416,
$(n=6)$	9363584	.... 9437056.

(*B.* 25, 6; *Schooten, Exercit. Math.* § 9.)

The number of prime numbers $> x$, is *very nearly* represented by $\dfrac{x}{\log_e x-1{,}08366}$. (*Leg.* 389—96.)

QUADRATIC FORMS OF PRIME NUMBERS.

(73.) Every prime number

$\pitchfork\ 4x+1$	is also $\pitchfork\ y^2+x^2;$
$\pitchfork\ 6x+1$	...... $\pitchfork\ y^2+3x^2;$
$\pitchfork\ 8x+1$	...... $\pitchfork\ y^2+x^2$, and $y^2\pm 2x^2;$
$\pitchfork\ 8x+3$	...... $\pitchfork\ y^2+2x^2;$
$\pitchfork\ 8x+7$	...... $\pitchfork\ y^2-2x^2;$
$\pitchfork\ 12x\pm 1$	...... $\pitchfork\ x^2\sim 3x^2;$
$\pitchfork\ 14x+1,\ +9,\ +11$	...... $\pitchfork\ y^2+7x^2;$
$\pitchfork\ 20x\pm 1,\ \pm 9$	...... $\pitchfork\ y^2-5x^2;$
$\pitchfork\ 20x+1,\ +9$	...... $\pitchfork\ y^2+5x^2;$
$\pitchfork\ 24x\pm 1,\ \pm 5$	...... $\pitchfork\ y^2\sim 6x^2.$
&c.	&c.

Every prime number $\nleftarrow y^2 + az^2$ is contained only once in that form.

If c is a prime number $\nleftarrow y^2 + az^2$, and b is contained m times in the form $t^2 + au^2$, bc is contained $2m$ times in that form, unless $b \overset{\cdot}{\underset{\cdot}{\hookleftarrow}} c$.

Every prime number $\nleftarrow ay^2 + bz^2$, or $\nleftarrow ay^2 + 2byz + 2ez^2$, is contained only once in that form.

If a, b, and c, are odd numbers, and unequal, no prime number can be contained more than once in the form

$$ay^2 + byz + cz^2.$$

If c is a prime number, any number $a \not> \frac{3}{4}c$ is contained only once in the form $\quad t^2 + cu^2$.

(*B*. Ch. IX; *Leg*. 142—7, 229—45.)

RESOLUTION OF NUMBERS INTO SQUARES.

(74.) $(x_1^2 + y_1^2)(x_2^2 + y_2^2)\dots(x_n^2 + y_n^2) \nleftarrow x^2 + y^2$; and x and y have 2^{n-1} different values.

$(x_1^2 + x_2^2 + x_3^2)(y_1^2 + y_2^2) \nleftarrow z_1^2 + z_2^2 + z_3^2 + z_4^2$.

$(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2) \nleftarrow z_1^2 + z_2^2 + z_3^2 + z_4^2$.

$(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2) \nleftarrow z_1^2 + z_2^2 + z_3^2 + z_4^2$; and generally,

$(x_1^2 - ax_2^2 - bx_3^2 + abx_4^2)(y_1^2 - ay_2^2 - by_3^2 + aby_4^2)$

$\quad \nleftarrow z_1^2 - az_2^2 - bz_3^2 + abz_4^2$. (*Lagr. Mem. Berl.* 1770.)

$x_1^2 - (a^2 + 1)x_2^2 \nleftarrow (a^2 + 1)y_1^2 - y_2^2$.

$(x_1^2 + ax_1y_1 + by_1^2)(x_2^2 + ax_2y_2 + by_2^2) \nleftarrow x^2 + axy + by^2$.

$(x_1^2 + ax_1y_1 + by_1^2)\dots(x_n^2 + ax_ny_n + by_n^2) \nleftarrow x^2 + axy + by^2$.

The equation $\quad x_1^2 + x_2^2 + x_3^2 + x_4^2 = ay$,
in which a is a prime number, is always possible.

Every integer is a square, or the sum of 2, 3, or 4, squares.

$$8ax+(a+2)(a-2)^2 \mathrel{⋕} (2ax_1-a+2)^2$$
$$+(2ax_2-a+2)^2+\&c.+(2ax_{a+2}-a+2)^2.$$

(*B*. Ch. vii; *Leg*. 148—55.)

QUADRATIC DIVISORS.

(75.) If t is prime to u, every divisor of the formula

$$t^2+au^2,$$

is also a divisor of x^2+a.

Every divisor of $t^2+au^2 \mathrel{⋕} a_1y^2+2b_1yz+c_1z^2$,

in which $a_1c_1-b_1{}^2=a$, and $b_1 \ngtr \frac{1}{2}a_1$, or $\frac{1}{2}c_1$; or $\ngtr \left(\frac{a}{5}\right)^{\frac{1}{2}}$.

Every divisor of $t^2\pm 2u^2 \mathrel{⋕} x^2\pm 2y^2$.

Every odd divisor of $t^2+3u^2 \mathrel{⋕} x^2\pm 3y^2$,

.... $t^2-5u^2 \mathrel{⋕} x^2-5y^2$.

(*B*. 103—9; *Leg*. 136—41.)

Let $R(a^{\frac{1}{2}(c-1)}\div c)$ represent the remainder of $a^{\frac{1}{2}(c-1)}\div c$, c being a prime number.

$$R(2^{\frac{1}{2}(c-1)}\div c)=+1, \text{ if } c \mathrel{⋕} 8n\pm 1,$$
$$R(2^{\frac{1}{2}(c-1)}\div c)=-1, \text{ if } c \mathrel{⋕} 8n\pm 3.$$

Every prime divisor, p, of $a^n+1 \mathrel{⋕} 2nx+1$; or at least is of the same form as the divisors of a^r+1, r being the quotient of n divided by an odd number.

Every prime divisor, p, of $a^n-1 \mathrel{⋕} 2nx+1$, or is of the same form as the divisors of a^r-1, r being any submultiple of n.

If m and n are prime numbers, then

$$R(n^{\frac{1}{2}(m-1)}\div m)=(-1)^{\frac{1}{4}(m-1)(n-1)}R(m^{\frac{1}{2}(n-1)}\div n).$$

The values of x in the equation

$$R(x^{\frac{1}{2}(c-1)}\div c)=\pm 1$$

are the remainders of any odd squares, $< 2c-1$, $\div c$.

If c is a prime number $\mathrel{⇹} 4n+1$, and a an odd divisor of x^2+cy^2, then
$$R(a^{\frac{1}{2}(c-1)} \div c) = \pm 1;$$
$+$ or $-$, according as $a \mathrel{⇹} 4n \pm 1$: if $c \mathrel{⇹} 4n+3$,
$$R(a^{\frac{1}{2}(c-1)} \div c) = 1.$$

If $c \mathrel{⇹} 4n+1$, and a is a divisor $\mathrel{⇹} 8n+1$, or $8n+3$, and b a divisor $\mathrel{⇹} 8n+5$, or $8n+7$, of the formula x^2+2cy^2; or if $c \mathrel{⇹} 4n+3$, and $a \mathrel{⇹} 8n \pm 1$, $b \mathrel{⇹} 8n \pm 3$, then
$$R(a^{\frac{1}{2}(c-1)} \div c) = 1, \quad \text{and } R(b^{\frac{1}{2}(c-1)} \div c) = -1.$$

No prime divisor of t^2+au^2 can be expressed by more than one quadratic divisor of that form. (*Leg.* 148—99, 232.)

Properties of the quadratic divisors of t^2+au^2, a being a prime number $\mathrel{⇹} 8n+1$. (*Leg.* 372—80.)

Every quadratic divisor of t^2+au^2 contains at least *one* number prime to a, or to $\frac{1}{2}a$, and $< a$. (*Leg.* 410—5.)

Further investigation of the linear and quadratic divisors of t^2+cu^2. (*Leg.* 204—12.)

Tables of linear and quadratic divisors explained. (*Leg.* 213—25.)

Application of the above tables to determine

[1] a prime number greater than a given number;

[2] whether a given number is prime or not. (*Leg.* 246—59.)

TERNARY DIVISORS.

(76.) No number $\mathrel{⇹} 4n$, or $8n+7$, can be of a ternary form.

If $c \mathrel{⇹} 4n+1$, or $4n+2$, the formula t^2+cu^2 has at least *one* ternary divisor.

If $c \mathrel{⇹} 8n+3$, the formula t^2+cu^2 has at least *one* quadratic divisor a, such that either a, or $2a$ is of a ternary form. (*Leg.* 260—6.)

If a quadratic divisor of t^2+cu^2 can be decomposed into
$$(a_1y+b_1z)^2+(a_2y+b_2z)^2+(a_3y+b_3z)^2,$$
then $c = (a_1b_2-a_2b_1)^2+(a_1b_3-a_3b_1)^2+(a_2b_3-a_3b_2)^2.$

To find the ternary divisor corresponding to

$$c = F^2 + G^2 + H^2:$$

Let F, G, H, be respectively resolved into fk_1k_2, gk_1k_3, hk_2k_3,

fk_2 and gk_3, fk_1 and hk_3, gk_1 and hk_2,

being respectively prime to each other.

Assume $$a_1b_2 - a_2b_1 = f;$$

$$fa_1 + ga_2 + ha_3 = 0,$$

$$fb_1 + gb_2 + hb_3 = 0;$$

the required divisor is

$$\overline{k_1(a_1y + b_1x)}\,|^2 + \overline{k_2(a_2y + b^2x)}\,|^2 + \overline{k_3(a_3y + b_3x)}\,|^2.$$

Since $k_1(a_1y + b_1x) + k_2(a_2y + b_2x) + k_3(a_3y + b_3x) = 0$, this divisor may be reduced by elimination to the form

$$ay^2 + byx + ex^2.$$

If $a = e$, $b = a$ or e, or $e = 0$, and at the same the least of the quantities a and $e > 2$, this divisor will have *two* ternary forms, and no more: if these conditions are not fulfilled, there will be only *one* ternary form, corresponding to any given value of c.

If either c, or $\frac{1}{2}c$, is a prime number,

[1] two different ternary forms of c cannot correspond to the same ternary divisor of $t^2 + au^2$;

[2] the formula $t^2 + au^2$ will have as many ternary divisors as there are ternary forms of the number c;

[3] each ternary divisor of $t^2 + au^2$ has only one ternary form.

If the number a is comprised in a ternary divisor $t^2 + cu^2$, c will be comprised in a ternary divisor of $t^2 + au^2$; and the corresponding ternary values of a and c will be the same.

If $py^2 + 2qyx + rx^2$, a quadratic divisor of $t^2 + cu^2$, has several ternary forms, and in these forms given numbers are substituted for x and y, the results will all be different, if each of them $< \frac{4}{3}c$.

Suppose p prime to c; then if c is a divisor of $t^2 + pu^2$, c will also be a divisor $t^2 + bu^2$, b being any number contained in the formula $py^2 + 2qyz + rz^2$.

Properties of *reciprocal* divisors. (*Leg.*278—313.)

SCALES OF NOTATION.

(77.) Every number may be represented by

$$a_n r^n + a_{n-1} r^{n-1} + a_{n-2} r^{n-2} + \&c. + a_1 r + a,$$

the radix r being any number, and a_n, &c. integers $< r$.

To transform a number from one scale to another: divide the number in its own scale by the new radix, and repeat the same process with that and all succeeding quotients; the remainders taken in order, beginning with the last, will be the digits which express the given number in the new scale.

In any scale, radix r, if the number itself and the sum of its digits be respectively divided by $r - 1$, the remainders will be the same.

If the sums of the odd and of the even digits be severally divided by $r + 1$, the difference of these remainders will equal the remainder of the number itself divided by $r + 1$.

Any number $\div 2^n$, if the number represented by the n last digits $\div 2^n$.

If the dimensions of any figure be expressed in feet, inches, and $\frac{1}{12}$th parts, the area or solidity may be most readily found by transforming the number of feet into the duodenary scale, and multiplying together the results in that scale; observing the same rules as in the multiplication of decimals.

Every number $< 2^{n+1}$ is the sum of some terms of the series $1,\ 2,\ 2^2,\ 2^3,$ &c. 2^n.

Every number $< 3^{n+1}$ may be expressed by the sums or differences of some terms of the series

$1,\ 3,\ 3^2,\ 3^3,$ &c. 3^n. (*B.* 121—9.)

TRIGONOMETRY.

(1.) *Divisions of the circle.* The circle has usually been divided into four quadrants;

each quadrant into 90 *degrees*, (90°);
each degree into 60 *minutes*, ($60'$);
each minute into 60 *seconds*, ($60''$).

The more minute divisions are generally expressed in decimal parts of $1''$; they are however sometimes expressed by dividing each second into 60 *thirds* ($60'''$), and so on.

The French have recently divided the quadrant into 100 *grades* (100^{g});

each grade into 100 *minutes*, ($100^{\backprime}$);
each minute into 100 *seconds*, ($100^{\backprime\backprime}$); and so on.

In Astronomy the circle is sometimes divided into 12 *signs*, each of which contains 30°.

(2.) To reduce $a^{g}\ b^{\backprime}\ c^{\backprime\backprime}$ to the English scale.

$A^{\circ} =$ *integral part of* $(a^{g}\ b^{\backprime}\ c^{\backprime\backprime} - 0{,}1 \times a^{g}\ b^{\backprime}\ c^{\backprime\backprime})$,
$B' =$ *integral part of* 0,6 (*preceding remainder*),
$C'' =$ *integral part of* 0,6 (*last remainder*).

$1^{g} = 54'$.	$1^{\circ} = 1^{g}\ 11^{\backprime}\ 11^{\backprime\backprime},\dot{1}$.
$1^{\backprime} = 32'',4$.	$1' = 1^{\backprime}\ \dot{8}\dot{5}^{\backprime\backprime},185$.
$1^{\backprime\backprime} = 0'',324$.	$1'' = 3^{\backprime\backprime},08641$ &c.

The arc equal in length to the radius

$= 57^{\circ}\ 17'\ 44''\ 48'''$ &c. $= 57^{\circ},22577$ &c. $= 206265''$ *nearly*.
$= 63^{g},66197$ &c.

(*W.* 19; *L.* 1—12; *C.* 631; *Enc. Met.* Vol. I, p. 672.)

(3.) To adapt any formula in which radius is unity to the general radius, r: multiply each term by such a power of r as shall render it of the same dimensions as the highest mentioned in the given formula. (*L.* 24; *W.* 18.)

(4.) *Mutual relations of the trigonometrical lines.*

$(\sin a)^2 + (\cos a)^2 = 1.$

$\dfrac{\sin a}{\cos a} = \tan a.$

$\dfrac{\cos a}{\sin a} = \cot a.$

$\tan a . \cot a = 1.$

$\sec a . \cos a = 1.$

$\operatorname{cosec} a . \sin a = 1.$

$1 + (\tan a)^2 = (\sec a)^2.$

$1 + (\cot a)^2 = (\operatorname{cosec} a)^2.$

$\operatorname{vers} a = 1 - \cos a.$

$\operatorname{covers} a = 1 - \sin a.$

$\operatorname{suvers} a = 1 + \cos a.$

(*L.* 35.)

(5.) The sine is positive in the *first* and *second* quadrants, and negative in the *third*, and *fourth*; the cosine is positive in the *first* and *fourth*, and negative in the *second* and *third* quadrants: from these data the signs of all other trigonometrical lines may be determined.

$\sin 2n\pi = 0, \qquad \cos 2n\pi = 1,$

$\tan 2n\pi = 0, \qquad \cot 2n\pi = \infty,$

$\sec 2n\pi = 1, \qquad \operatorname{cosec} 2n\pi = \infty.$

$\sin (2n + \tfrac{1}{2})\pi = 1, \qquad \cos (2n + \tfrac{1}{2})\pi = 0,$

$\tan (2n + \tfrac{1}{2})\pi = \infty, \qquad \cot (2n + \tfrac{1}{2})\pi = 0,$

$\sec (2n + \tfrac{1}{2})\pi = \infty, \qquad \operatorname{cosec} (2n + \tfrac{1}{2})\pi = 1.$

$\sin (2n + 1)\pi = 0, \qquad \cos (2n + 1)\pi = -1,$

$\tan (2n + 1)\pi = 0, \qquad \cot (2n + 1)\pi = -\infty,$

$\sec (2n + 1)\pi = -1, \qquad \operatorname{cosec} (2n + 1)\pi = \infty.$

$\sin (2n + \tfrac{3}{2})\pi = -1, \qquad \cos (2n + \tfrac{3}{2})\pi = 0,$

$\tan (2n + \tfrac{3}{2})\pi = -\infty, \qquad \cot (2n + \tfrac{3}{2})\pi = 0,$

$\sec (2n + \tfrac{3}{2})\pi = \infty, \qquad \operatorname{cosec} (2n + \tfrac{3}{2})\pi = -1.$

(6.) *Different arcs whose sine, or cosine is the same.*

$$\sin(2n\pi \pm a) = \sin(\overline{2n+1}.\pi \mp a),$$
$$= -\sin(\overline{2n+1}.\pi \pm a);$$
$$\sin(2n\pi + a) = -\sin(2n\pi - a),$$
$$\sin(\overline{2n+1}.\pi - a) = -\sin(\overline{2n+1}.\pi + a).$$

$$\cos(2n\pi + a) = \cos(2n\pi - a),$$
$$\cos(\overline{2n+1}.\pi - a) = \cos(\overline{2n+1}.\pi + a),$$
$$\cos(2n\pi \pm a) = -\cos(\overline{2n+1}.\pi \pm a),$$
$$\cos(+a) = \cos(-a),$$
$$\sin(+a) = -\sin(-a).$$

$$\sin(\overline{2n+\tfrac{1}{2}}.\pi \pm a) = \cos(2n\pi \pm a),$$
$$\cos(\overline{2n+\tfrac{1}{2}}.\pi \mp a) = \mp\sin(2n\pi + a),$$
$$\sin(\overline{2n+\tfrac{3}{2}}.\pi \pm a) = -\cos(2n\pi \pm a),$$
$$\cos(\overline{2n+\tfrac{3}{2}}.\pi \pm a) = \pm\sin(2n\pi + a).$$
$$\sin(\tfrac{1}{2}\pi \pm a) = \cos a,$$
$$\cos(\tfrac{1}{2}\pi \pm a) = \pm\sin a.$$

L. 36—41; *C.* 64—81.)

(7.) *Values of sin* a.

$\cos a . \tan a.$

$\dfrac{\cos a}{\cot a}.$

$\overline{1-(\cos a)^2}|^{\frac{1}{2}}.$

$\overline{1+(\cot a)^2}|^{-\frac{1}{2}}.$

$\tan a . \overline{1+(\tan a)^2}|^{-\frac{1}{2}}.$

$2\sin\tfrac{1}{2}a . \cos\tfrac{1}{2}a.$

$\overline{\tfrac{1}{2}\{1-(\cos 2a)^2\}}|^{\frac{1}{2}}.$

$\dfrac{2\tan\frac{1}{2}a}{1+(\tan\frac{1}{2}a)^2}.$

$2.\overline{\cot\tfrac{1}{2}a + \tan\tfrac{1}{2}a}|^{-\frac{1}{2}}.$

$\tfrac{1}{3}\sqrt{3}\{\sin(30^0+a)-\sin(30^0-a)\}.$

$2.\overline{\sin(45^0+\tfrac{1}{2}a)}|^2 - 1.$

$1-2.\overline{\sin(45^0-\tfrac{1}{2}a)}|^2.$

$\dfrac{1-\overline{\tan(45^0-\frac{1}{2}a)}|^2}{1+\overline{\tan(45^0-\frac{1}{2}a)}|^2}.$

$\dfrac{\tan(45^0+\frac{1}{2}a)-\tan(45^0-\frac{1}{2}a)}{\tan(45^0+\frac{1}{2}a)+\tan(45^0-\frac{1}{2}a)}.$

$\sin(60^0+a)-\sin(60^0-a).$

(8.) *Values of cos* a.

$$\frac{\sin a}{\tan a}.$$

$$\sin a \,.\, \cot a.$$

$$\left.\overline{1-(\sin a)^2}\right|^{\frac{1}{2}}.$$

$$\left.\overline{1+(\tan a)^2}\right|^{-\frac{1}{2}}.$$

$$\cot a \,.\, \left.\overline{1+(\cot a)^2}\right|^{-\frac{1}{2}}.$$

$$(\cos \tfrac{1}{2}a)^2-(\sin \tfrac{1}{2}a)^2.$$

$$1-2\,(\sin \tfrac{1}{2}a)^2.$$

$$2\,(\cos \tfrac{1}{2}a)^2-1.$$

$$\left.\overline{\tfrac{1}{2}(1+\cos 2a)}\right|^{\frac{1}{2}}.$$

$$\frac{1-(\tan \frac{1}{2}a)^2}{1+(\tan \frac{1}{2}a)^2}.$$

$$\frac{\cot \frac{1}{2}a-\tan \frac{1}{2}a}{\cot \frac{1}{2}a+\tan \frac{1}{2}a}.$$

$$\frac{1}{1+\tan a \,.\, \tan \frac{1}{2}a}.$$

$$\frac{2}{\tan(45^\circ+\frac{1}{2}a)+\cot(45^\circ+\frac{1}{2}a)}.$$

$$2\cos(45^\circ+\tfrac{1}{2}a)\cos(45^\circ-\tfrac{1}{2}a).$$

$$\cos(60^\circ+a)+\cos(60^\circ-a).$$

(9.) *Values of tan* a.

$$\frac{\sin a}{\cos a}.$$

$$\frac{1}{\cot a}.$$

$$\left.\overline{\frac{1}{(\cos a)^2}-1}\right|^{\frac{1}{2}}.$$

$$\sin a \,.\, \left.\overline{1-(\sin a)^2}\right|^{-\frac{1}{2}}.$$

$$\frac{\left.\overline{1-(\cos a)^2}\right|^{\frac{1}{2}}}{\cos a}.$$

$$\frac{2\tan \frac{1}{2}a}{1-(\tan \frac{1}{2}a)^2}.$$

$$\frac{2\cot \frac{1}{2}a}{(\cot \frac{1}{2}a)^2-1}.$$

$$\frac{2}{\cot \frac{1}{2}a-\tan \frac{1}{2}a}.$$

$$\cot a-2\cot 2a.$$

$$\frac{1-\cos 2a}{\sin 2a}.$$

$$\frac{\sin 2a}{1+\cos 2a}.$$

$$\left.\overline{\frac{1-\cos 2a}{1+\cos 2a}}\right|^{\frac{1}{2}}.$$

$$\tfrac{1}{2}\left\{\tan(45^\circ+\tfrac{1}{2}a)-\tan(45^\circ-\tfrac{1}{2}a)\right\}.$$

(10.) *Formulæ relating to two Arcs.*

$$\sin(a \pm b)=\sin a \,.\, \cos b \pm \cos a \,.\, \sin b.$$

$$\cos(a \pm b)=\cos a \,.\, \cos b \mp \sin a \,.\, \sin b.$$

$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \,.\, \tan b}.$$

$$\cot(a \pm b) = \frac{\cot a \,.\, \cot b \mp 1}{\cot b \pm \cot a}.$$

$$\left.\begin{array}{l}\sin(45^0 \pm b)\\ \cos(45^0 \mp b)\end{array}\right\} = \frac{1}{\sqrt{2}}(\cos b \pm \sin b).$$

$$\tan(45^0 \pm b) = \frac{1 \pm \tan . b}{1 \mp \tan . b}.$$

$$\tan(45^0 \pm \tfrac{1}{2}b) = \frac{1 \pm \sin b}{\cos b} = \frac{\cos b}{1 \mp \sin b}.$$

$$\tan(45^0 \pm b) = \left.\frac{1 \pm \sin b}{1 \mp \sin b}\right|^{\frac{1}{2}}.$$

$$\frac{\sin(a+b)}{\sin(a-b)} = \frac{\tan a + \tan b}{\tan a - \tan b} = \frac{\cot b + \cot a}{\cot b - \cot a}.$$

$$\frac{\cos(a+b)}{\cos(a-b)} = \frac{\cot b - \tan a}{\cot b + \tan a} = \frac{\cot a - \tan b}{\cot a + \tan b}.$$

$$\frac{\sin a + \sin b}{\sin a - \sin b} = \frac{\tan \frac{1}{2}(a+b)}{\tan \frac{1}{2}(a-b)}.$$

$$\frac{\cos b + \cos a}{\cos b - \cos a} = \frac{\cot \frac{1}{2}(a+b)}{\tan \frac{1}{2}(a-b)}.$$

$$\frac{\sin a + \sin b}{\cos a + \cos b} = \frac{\cos b - \cos a}{\sin a - \sin b} = \tan \tfrac{1}{2}(a+b).$$

$$\sin a . \cos b = \tfrac{1}{2}\sin(a+b) + \tfrac{1}{2}\sin(a-b).$$

$$\cos a . \sin b = \tfrac{1}{2}\sin(a+b) - \tfrac{1}{2}\sin(a-b).$$

$$\sin a . \sin b = \tfrac{1}{2}\cos(a-b) - \tfrac{1}{2}\cos(a+b).$$

$$\cos a . \cos b = \tfrac{1}{2}\cos(a+b) + \tfrac{1}{2}\cos(a-b).$$

$$\sin a + \sin b = 2\sin \tfrac{1}{2}(a+b) . \cos \tfrac{1}{2}(a-b).$$

$$\cos a + \cos b = 2\cos \tfrac{1}{2}(a+b) . \cos \tfrac{1}{2}(a-b).$$

$$\sin a - \sin b = 2\sin \tfrac{1}{2}(a-b) . \cos \tfrac{1}{2}(a+b).$$

$$\cos b - \cos a = 2\sin \tfrac{1}{2}(a-b) . \sin \tfrac{1}{2}(a+b).$$

$$\tan a \pm \tan b = \frac{\sin(a \pm b)}{\cos a . \cos b}.$$

$$\cot b \pm \cot a = \frac{\sin(a \pm b)}{\sin a . \sin b}.$$

$$\left.\begin{array}{l}(\sin a)^2 - (\sin b)^2 \\ (\cos b)^2 - (\cos a)^2\end{array}\right\} = \sin(a+b) . \sin(a-b).$$

$$\left.\begin{array}{l}(\cos a)^2 - (\sin b)^2 \\ (\cos b)^2 - (\sin a)^2\end{array}\right\} = \cos(a+b) . \cos(a-b).$$

$$\tan a - \tan b = \frac{\sin(a+b) . \sin(a-b)}{\cos a . \cos b}.$$

$$\cot b - \cot a = \frac{\sin(a+b) . \sin(a-b)}{\sin a . \sin b}.$$

$$\sin b = \sin(a+b) . \cos a - \cos(a+b) . \sin a.$$

$$\cos b = \sin(a+b) . \sin a + \cos(a+b) . \cos a.$$

$$\sin(n+2)\,a + \sin na = 2 \sin(n+1)\,a . \cos a.$$

$$\sin(n+2)\,a - \sin na = 2 \cos(n+1)\,a . \sin a.$$

$$\cos na + \cos(n+2)\,a = 2 \cos(n+1)\,a . \cos a.$$

$$\cos na - \cos(n+2)\,a = 2 \sin(n+1)\,a . \sin a.$$

$$\sin(n+2)\,a . \sin na = \overline{\sin(n+1)\,a}\,\rvert^2 - (\sin a)^2.$$

(*L.* 43—54; *W.* Ch. ii.)

(11.) *Formulæ relating to double arcs.*

$$\sin 2a = 2 \sin a . \cos a.$$

$$\begin{aligned}\cos 2a &= (\cos a)^2 - (\sin a)^2; \\ &= 2(\cos a)^2 - 1; \\ &= 1 - 2(\sin a)^2.\end{aligned}$$

$$\begin{aligned}\tan 2a &= \frac{2 \tan a}{1 - (\tan a)^2}; \\ &= \frac{2 \cot a}{(\cot a)^2 - 1};\end{aligned}$$

$$\tan 2a = \frac{2}{\cot a - \tan a}.$$

$$\begin{aligned}\cot 2a &= \frac{(\cot a)^2 - 1}{2 \cot a}; \\ &= \tfrac{1}{2}(\cot a - \tan a).\end{aligned}$$

$$\begin{aligned}\sec 2a &= \frac{(\sec a)^2}{1 - (\tan a)^2}; \\ &= \frac{1 + (\tan a)^2}{1 - (\tan a)^2};\end{aligned}$$

$$\sec 2a = \frac{(\operatorname{cosec} a)^2}{(\cot a)^2 - 1}.$$

$$\operatorname{cosec} 2a = \frac{(\operatorname{cosec} a)^2}{2 \cot a};$$

$$\operatorname{cosec} 2a = \frac{(\cot a)^2 + 1}{2 \cot a};$$

$$= \tfrac{1}{2}(\tan a + \cot a).$$

$$\sin \tfrac{1}{2} a = \left.\overline{\tfrac{1}{2}(1 - \cos a)}\right|^{\frac{1}{2}}.$$

$$\cos \tfrac{1}{2} a = \left.\overline{\tfrac{1}{2}(1 + \cos a)}\right|^{\frac{1}{2}}.$$

$$\tan \tfrac{1}{2} a = \frac{\sin a}{1 + \cos a};$$

$$= \frac{1 - \cos a}{\sin a};$$

$$= \left.\overline{\frac{1 - \cos a}{1 + \cos a}}\right|^{\frac{1}{2}};$$

$$= \frac{\tan a}{\sec a + 1};$$

$$= \frac{\sec a - 1}{\tan a};$$

$$= \left.\overline{\frac{\sec a - 1}{\sec a + 1}}\right|^{\frac{1}{2}};$$

$$= \operatorname{cosec} a - \cot a.$$

$$\cot \tfrac{1}{2} a = \frac{1 + \cos a}{\sin a};$$

$$= \frac{\sin a}{1 - \cos a};$$

$$= \left.\overline{\frac{1 + \cos a}{1 - \cos a}}\right|^{\frac{1}{2}};$$

$$= \frac{\sec a + 1}{\tan a};$$

$$= \left.\overline{\frac{\sec a + 1}{\sec a - 1}}\right|^{\frac{1}{2}};$$

$$= \operatorname{cosec} a + \cot a.$$

$$\sec \tfrac{1}{2} a = \left.\overline{\frac{2 \sec a}{\sec a + 1}}\right|^{\frac{1}{2}};$$

$$\operatorname{cosec} \tfrac{1}{2} a = \left.\overline{\frac{2 \sec a}{\sec a - 1}}\right|^{\frac{1}{2}};$$

(*L.* 55—61.)

(12.) *Values of the sine, cosine, &c. of* 30°, 45°, *and* 60°.

$\sin 30^\circ = \cos 60^\circ = \frac{1}{2}$, $\cos 30^\circ = \sin 60^\circ = \frac{1}{2}\sqrt{3}$,

$\tan 30^\circ = \cot 60^\circ = \frac{1}{3}\sqrt{3}$, $\cot 30^\circ = \tan 60^\circ = \sqrt{3}$,

$\sec 30^\circ = \operatorname{cosec} 60^\circ = \frac{2}{3}\sqrt{3}$, $\operatorname{cosec} 30^\circ = \sec 60^\circ = 2$.

$\sin 45^\circ = \cos 45^\circ = \frac{1}{2}\sqrt{2}$, $\tan 45^\circ = \cot 45^\circ = 1$,

$\sec 45^\circ = \operatorname{cosec} 45^\circ = \sqrt{2}$.

(13.) *Formulæ relating to three arcs.*

$$\sin(a+b+c) = \sin a.\cos b.\cos c + \cos a.\sin b.\cos c$$
$$+ \cos a.\cos b.\sin c - \sin a.\sin b.\sin c.$$

$$\cos(a+b+c) = \cos a.\cos b.\cos c - \cos a.\sin b.\sin c$$
$$- \sin a.\cos b.\sin c - \sin a.\sin b.\cos c.$$

$$\tan(a+b+c) = \frac{\tan a + \tan b + \tan c - \tan a.\tan b.\tan c}{1 - (\tan a.\tan b + \tan a.\tan c + \tan b.\tan c)}.$$

(14.) *Relations between the sides and angles of plane triangles.*

Let A, B, C, be the angles,

a, b, c, the sides subtending them;

$s = \frac{1}{2}(a+b+c)$.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

$$\sin A = \frac{2}{bc}.\overline{s(s-a)(s-b)(s-c)}|^{\frac{1}{2}}.$$

$$(\sin \tfrac{1}{2}A)^2 = \frac{(s-b)(s-c)}{bc}.$$

$$(\cos \tfrac{1}{2}A)^2 = \frac{s(s-a)}{bc}.$$

$$(\tan \tfrac{1}{2}A)^2 = \frac{(s-b)(s-c)}{s(s-a)}.$$

$$\sin 2A + \sin 2B + \sin 2C = \sin A.\sin B.\sin C.$$
$$\cos 2A + \cos 2B + \cos 2C = 4\cos A.\cos B.\cos C - 1.$$
$$\tan A + \tan B + \tan C = \tan A.\tan B.\tan C.$$

(*W. Ch.* ii; *L.* 365.)

The *area* $= \overline{s(s-a)(s-b)(s-c)}|^{\frac{1}{2}}$;

$$= \frac{2abc}{a+b+c}\cos\tfrac{1}{2}A.\cos\tfrac{1}{2}B.\cos\tfrac{1}{2}C.$$

Let R_1 be the radius of the inscribed, and R_2 that of the circumscribing circle; then

$$R_1 = \overline{\tfrac{1}{s}(s-a)(s-b)(s-c)}\Big|^{\frac{1}{2}};$$
$$= s.\tan\tfrac{1}{2}A.\tan\tfrac{1}{2}B.\tan\tfrac{1}{2}C,$$
$$R_2 = \tfrac{1}{4}abc.\overline{s(s-a)(s-b)(s-c)}\Big|^{-\frac{1}{2}}.$$

(*Legendre, Geom.* Note v; *Hind, Trig.* 153—64.)

(15.) *Values of the side c.*

$a\dfrac{\sin C}{\sin A}.$ *

$a\cos B + b\cos A.$

$\overline{a^2+b^2-2ab\cos C}\big|^{\frac{1}{2}}.$

$\overline{(a+b)^2-4ab(\cos\frac{1}{2}C)^2}\big|^{\frac{1}{2}}.$

$\overline{(a-b)^2+4ab(\sin\frac{1}{2}C)^2}\big|^{\frac{1}{2}}.$

$b\cos A \pm a.\overline{1-(\sin B)^2}\big|^{\frac{1}{2}}.$

$b\cos A \pm \overline{a^2-(b\sin A)^2}\big|^{\frac{1}{2}}.$

$\dfrac{a}{\cos B+\sin B.\cot C}.$

(16.) *Values of* $\sin C$.

$\sin(A+B).$

$\dfrac{c}{b}\sin B.$

$c\sin B.\overline{a^2+c^2-2ac\cos B}\big|^{\frac{1}{2}}.$

$\overline{1-\left(\dfrac{b^2+a^2-c^2}{2ab}\right)^2}\Bigg|^{\frac{1}{2}}.$

$\dfrac{\sin B}{b}\left\{a\cos B \pm \overline{b^2-(a\sin B)^2}\big|^{\frac{1}{2}}\right\}.$

(17.) *Values of* $\cos C$.

$-\cos(A+B).$

$\overline{1-\dfrac{c^2}{b^2}(\sin B)^2}\Bigg|^{\frac{1}{2}}.$

$\dfrac{a-c\cos B}{(a^2+c^2-2ac\cos B)^{\frac{1}{2}}}.$

$\dfrac{a(\sin B)^2 \mp \cos B.\overline{b^2-(a\sin B)^2}\big|^{\frac{1}{2}}}{b},$

$\dfrac{2s(s-c)}{ab}-1,$

(18.) *Values of* $\tan C$.

$-\tan(A+B).$

$c\sin B.\overline{b^2-(c\sin B)^2}\big|^{\frac{1}{2}}.$

$\dfrac{c\sin B}{a-c\cos B}.$

$\overline{\left(\dfrac{2ab}{a^2+b^2-c^2}\right)^2-1}\Bigg|^{\frac{1}{2}}.$

$\dfrac{a\cos B+\overline{b^2-(a\sin B)^2}\big|^{\frac{1}{2}}}{a\sin B-\cot B.\overline{b^2-(a\sin B)^2}\big|^{\frac{1}{2}}}.$

---

* Analogous formulæ may be obtained by substituting A, a, for B, b, and *vice versa*, in this, and several of the following formulæ.

(19.) *Right angled triangles: C, the right angle.*

[1] Given a, b: $\tan A = \frac{a}{b}$;

$$\log \operatorname{Tan} A^{*} = \log r + \log a - \log b.$$

$$c = (a^2 + b^2)^{\frac{1}{2}}; \text{ or}$$

$$c = \frac{b}{\cos A};$$

$$\log c = \log r + \log b - \log \operatorname{Cos} A.$$

[2] Given a, c: $b = (c^2 - a^2)^{\frac{1}{2}}$,

$$\log b = \tfrac{1}{2} \log (c + a) + \tfrac{1}{2} \log (c - a).$$

$$\cos A = \frac{b}{c};$$

$$\log \operatorname{Cos} A = \log b - \log c + \log r.$$

$$\sin B = \cos A.$$

[3] Given A, c: $a = c \sin A$;

$$\log a = \log c + \log \operatorname{Sin} A - \log r.$$

$$B = 90^\circ - A.$$

$$b = c \cos A;$$

$$\log b = \log c + \log \operatorname{Cos} A - \log r.$$

or b may be determined as above.

[4] Given A, b: $B = 90^\circ - A$.

$$a = b \tan A;$$

$$\log a = \log b + \log \operatorname{Tan} A - \log r.$$

$$c = \frac{b}{\cos A};$$

$$\log c = \log r + \log b - \log \operatorname{Cos} A;$$

(*W. Ch.* v; *L.* 67—71; *Leg.* 48—52; *C.* 529—58.)

---

[*] The sine, cosine, &c. to the tabular radius r will, for the sake of distinction, be denoted by capital letters. See (*Enc. Met. Art. Trigonom.*)

(20.) *Solution of oblique angled plane triangles.*

[1] Given A, B, a;

$$C = 180° - (A + B).$$

$$b = a\frac{\sin B}{\sin A};$$

$$\log b = \log a + \log \text{Sin } B - \log \text{Sin } A.$$

c may be similarly determined.

[2] Given B, a, b;

$$\sin A = \frac{a}{b} \sin B;$$

$$\log \text{Sin } A = \log a - \log b + \log \text{Sin } B.$$

This result is ambiguous if $a > b$, and B is an acute angle.

$$C = 180° - (A + B).$$

$$c = b\frac{\sin C}{\sin B};$$

$$\log c = \log b + \log \text{Sin } C - \log \text{Sin } B:$$

$$\text{or } c = a \cos B \pm \overline{b^2 - (a \sin B)^2}|^{\frac{1}{2}}.$$

[3] Given C, a, b;

$$\tan \tfrac{1}{2}(A - B) = \frac{a-b}{a+b} \tan \tfrac{1}{2}(A + B);$$

$$\log \text{Tan } \tfrac{1}{2}(A-B) = \log(a-b) - \log(a+b) + \log \text{Tan } \tfrac{1}{2}(A+B),$$

$$A + B = 180° - C:$$

or thus: suppose $a > b$, and let $\tan \theta = \frac{a}{b}$;

$\log \text{Tan } \theta = \log r + \log a - \log b$: then

$$\tan \tfrac{1}{2}(A - B) = \tan \tfrac{1}{2}(A + B) \,.\, \tan(\theta - 45°),$$

$$\log \text{Tan } \tfrac{1}{2}(A-B) = \log \text{Tan } \tfrac{1}{2}(A+B) + \log \text{Tan}(\theta - 45°) - \log r.$$

knowing $A + B$, and $A - B$, A and B are determined.

$$c = a\,\frac{\sin C}{\sin A};$$

$\log c = \log a + \log \text{Sin } C - \log \text{Sin } A.$

or $c = (a^2 + b^2 - 2ab\cos C)^{\frac{1}{2}}$:

assume $(\tan\theta)^2 = \dfrac{2ab}{(a-b)^2}\,.\,\text{vers } C;$

$\log \text{Tan }\theta = \frac{1}{2}\{\log 2 + \log a + \log b + \log \text{Vers } C + \log r\} - \log(a-b);$

then $c = \dfrac{a-b}{\cos\theta};$

$\log c = \log(a-b) + \log r - \log \text{Cos }\theta.$

or assume $(\sin\theta)^2 = \dfrac{2ab}{(a+b)^2}\,.\,(1+\cos C),$

$\log \text{Sin }\theta = \frac{1}{2}\{\log 2 + \log a + \log b + \log(r + \text{Cos } C) + \log r\} - \log(a+b);$

then $c = (a+b)\cos\theta;$

$\log c = \log(a+b) + \log \text{Cos }\theta - \log r.$

[4] Given a, b, c.

First Method: $\sin A = \dfrac{2}{bc}\,.\,\overline{s(s-a)(s-b)(s-c)}|^{\frac{1}{2}};$

$\log \text{Sin } A = \log r + \log 2 - \log b - \log c$
$\qquad + \frac{1}{2}\{\log s + \log(s-a) + \log(s-b) + \log(s-c)\}.$

Second Method: $(\sin\frac{1}{2}A)^2 = \dfrac{(s-b)(s-c)}{bc};$

$\log \text{Sin }\frac{1}{2}A = \frac{1}{2}\{\log(s-b) + \log(s-c) - \log b - \log c\} + \log r.$

Third Method: $(\cos\frac{1}{2}A)^2 = \dfrac{s(s-a)}{bc};$

$\log \text{Cos }\frac{1}{2}A = \frac{1}{2}\{\log s + \log(s-a) - \log b - \log c\} + \log r.$

Fourth Method: $(\tan\frac{1}{2}A)^2 = \frac{(s-b)(s-c)}{s(s-a)}$;

$$\log \text{Tan} \tfrac{1}{2}A = \tfrac{1}{2}\{\log(s-b)+\log(s-c)-\log s-\log(s-a)\}+\log r.$$

If A *nearly* $= 90°$, the first method is inapplicable.

The second method is preferable if $A < 90°$; the third if $A > 90°$.

If A *nearly* $= 180°$, the fourth method is inapplicable.
(*W. Ch.* v; *L.* 72—84; *Leg.* 53—7; *C.* 559—615.)

SPHERICAL TRIGONOMETRY.

(21.) *General Principles.*

Every section of a sphere made by a plane is a Circle.

The distance between the poles of two great circles is equal to the inclination of their planes.

The pole of a great circle is the pole of all parallel small circles.

If the angle subtended by the arc a is invariable, then

$$a \propto \sin \textit{dist. from pole}.$$

If the intersections of three great circles are the poles of three others, the intersections of the latter will be the poles of the former.

The sum of the three sides is always $< 360°$.

The sum of the three angles is always $> 180°$, and $< 540°$.

The sum of any two sides is greater than the third side.

The angles at the base of an isosceles spherical triangle are equal; and conversely.

The greater side subtends the greater angle; and conversely.
(*W. Ch.* viii. *prop.* 1—12; *L.* 102—66; *C.* 960—1010.)

(22.) *Relations between the sides and angles of spherical triangles.*

Let $S = \frac{1}{2}(A + B + C)$,

$s = \frac{1}{2}(a + b + c)$; and

$N = \overline{-\cos S . \cos (S - A) . \cos (S - B) . \cos (S - C)}\Big|^{\frac{1}{2}}$,

$n = \overline{\sin s . \sin (s - a) . \sin (s - b) . \sin (s - c)}\Big|^{\frac{1}{2}}$.

$$\cos A = \frac{\cos a - \cos b . \cos c}{\sin b . \sin c}.$$

$$\cos a = \frac{\cos A + \cos B . \cos C}{\sin B . \sin C}.$$

$\sin b . \sin c . (\sin \frac{1}{2} A)^2 = \sin (s - b) . \sin (s - c)$.

$\sin b . \sin c . (\cos \frac{1}{2} A)^2 = \sin s . \sin (s - a)$.

$\sin A . \sin b . \sin c = 2n$.

$\sin (a + b) (\sin \frac{1}{2} C)^2 = \cos \frac{1}{2}(A + B) . \cos \frac{1}{2}(A - B) . \sin c$.

$\sin (a - b) (\cos \frac{1}{2} C)^2 = \sin \frac{1}{2}(A + B) . \sin \frac{1}{2}(A - B) . \sin c$.

$$\sin \tfrac{1}{2}(A + B) = \frac{\cos \frac{1}{2} C}{\cos \frac{1}{2} c} \cos \tfrac{1}{2}(a - b).$$

$$\sin \tfrac{1}{2}(A - B) = \frac{\cos \frac{1}{2} C}{\sin \frac{1}{2} c} \sin \tfrac{1}{2}(a - b).$$

$$\cos \tfrac{1}{2}(A + B) = \frac{\sin \frac{1}{2} C}{\cos \frac{1}{2} c} \cos \tfrac{1}{2}(a + b).$$

$$\cos \tfrac{1}{2}(A - B) = \frac{\sin \frac{1}{2} C}{\sin \frac{1}{2} c} \sin \tfrac{1}{2}(a + b).$$

$\sin B . \sin C . (\sin \frac{1}{2} a)^2 = -\cos S . \cos (S - A)$.

$\sin B . \sin C . (\cos \frac{1}{2} a)^2 = \cos (S - B) . \cos (S - C)$.

$\sin B . \sin C . \sin a = 2N$.

$\sin (A + B) (\cos \frac{1}{2} c)^2 = \cos \frac{1}{2}(a + b) . \cos \frac{1}{2}(a - b) . \sin C$.

$$\sin(A-B)(\sin\tfrac{1}{2}c)^2=\sin\tfrac{1}{2}(a+b)\,.\,\sin\tfrac{1}{2}(a-b)\,.\,\sin C.$$

$$\sin\tfrac{1}{2}(a+b)=\frac{\sin\frac{1}{2}c}{\sin\frac{1}{2}C}\cos\tfrac{1}{2}(A-B).$$

$$\sin\tfrac{1}{2}(a-b)=\frac{\sin\frac{1}{2}c}{\sin\frac{1}{2}C}\sin\tfrac{1}{2}(A-B).$$

$$\cos\tfrac{1}{2}(a+b)=\frac{\cos\frac{1}{2}c}{\sin\frac{1}{2}C}\cos\tfrac{1}{2}(A+B).$$

$$\cos\tfrac{1}{2}(a-b)=\frac{\cos\frac{1}{2}c}{\cos\frac{1}{2}C}\sin\tfrac{1}{2}(A+B).$$

$$\frac{\sin(a-b)}{\sin(a+b)}=(\tan\tfrac{1}{2}C)^2\,.\,\tan\tfrac{1}{2}(A+B)\,.\,\tan\tfrac{1}{2}(A-B).$$

$$\frac{\sin(A-B)}{\sin(A+B)}=(\cot\tfrac{1}{2}c)^2\,.\,\tan\tfrac{1}{2}(a+b)\,.\,\tan\tfrac{1}{2}(a-b).$$

$$\frac{\sin\frac{1}{2}(a-b)\,.\,\cot\frac{1}{2}c}{\cos\frac{1}{2}(a+b)}=\frac{\sin\frac{1}{2}(A-B)\,.\,\cot\frac{1}{2}C}{\cos\frac{1}{2}(A+B)}.$$

$$\sin A\,.\,\sin B-\sin a\,.\,\sin b=\cos A\,.\,\cos B\,.\,\cos c+\cos a\,.\,\cos b\,.\,\cos C.$$

$$(\sin C)^2\,.\,\sin(a+b)\,.\,\sin(a-b)=(\sin c)^2\,.\,\sin(A+B)\,.\,\sin(A-B).$$

$$\cos c=\cos(a-b)(\cos\tfrac{1}{2}C)^2+\cos(a+b)(\sin\tfrac{1}{2}C)^2.$$

$$(\sin\tfrac{1}{2}c)^2=\overline{\sin\tfrac{1}{2}(a-b)\,.\,\cos\tfrac{1}{2}C}\big|^2+\overline{\sin\tfrac{1}{2}(a+b)\,.\,\sin\tfrac{1}{2}C}\big|^2.$$

$$(\cos\tfrac{1}{2}c)^2=\overline{\cos\tfrac{1}{2}(a-b)\,.\,\cos\tfrac{1}{2}C}\big|^2+\overline{\cos\tfrac{1}{2}(a+b)\,.\,\sin\tfrac{1}{2}C}\big|^2.$$

$$(\tan\tfrac{1}{2}C)^2=\frac{\sin(a-b)}{\sin(a+b)}\cdot\frac{\cos B+\cos A}{\cos B-\cos A}.$$

$$\cos C=-\cos(A-B)(\sin\tfrac{1}{2}c)^2-\cos(A+B)(\cos\tfrac{1}{2}c)^2.$$

$$(\cos\tfrac{1}{2}C)^2=\overline{\sin\tfrac{1}{2}(A-B)\,.\,\sin\tfrac{1}{2}c}\big|^2+\overline{\sin\tfrac{1}{2}(A+B)\,.\,\cos\tfrac{1}{2}c}\big|^2.$$

$$(\sin\tfrac{1}{2}C)^2=\overline{\cos\tfrac{1}{2}(A-B)\,.\,\sin\tfrac{1}{2}c}\big|^2+\overline{\cos\tfrac{1}{2}(A+B)\,.\,\cos\tfrac{1}{2}c}\big|^2.$$

$$(\cot\tfrac{1}{2}c)^2=\frac{\sin(A-B)}{\sin(A+B)}\cdot\frac{\cos b+\cos a}{\cos b-\cos a}.$$

$$\sin(s-a)=\frac{N}{2\sin\tfrac{1}{2}A.\cos\tfrac{1}{2}B.\cos\tfrac{1}{2}C};$$

$$=2\frac{n}{N}\cos\tfrac{1}{2}A.\sin\tfrac{1}{2}B.\sin\tfrac{1}{2}C;$$

$$=(n.\cot\tfrac{1}{2}A.\tan\tfrac{1}{2}B.\tan\tfrac{1}{2}C)^{\frac{1}{2}}.$$

$$\cos(s-a)=\frac{1-\cos A+\cos B+\cos C}{4\sin\tfrac{1}{2}A.\cos\tfrac{1}{2}B.\cos\tfrac{1}{2}C}.$$

$$\cos(S-A)=\frac{n}{2\cos\tfrac{1}{2}a.\sin\tfrac{1}{2}b.\sin\tfrac{1}{2}c};$$

$$=2\frac{N}{n}\sin\tfrac{1}{2}a.\cos\tfrac{1}{2}b.\cos\tfrac{1}{2}c;$$

$$=(N.\tan\tfrac{1}{2}a.\cot\tfrac{1}{2}b.\cot\tfrac{1}{2}c)^{\frac{1}{2}}.$$

$$\sin(S-A)=\frac{1+\cos a-\cos b-\cos c}{4\cos\tfrac{1}{2}a.\sin\tfrac{1}{2}b.\sin\tfrac{1}{2}c}.$$

$$\sin\tfrac{1}{2}A.\sin\tfrac{1}{2}B.\sin\tfrac{1}{2}C=\frac{\sin(s-a).\sin(s-b).\sin(s-c)}{\sin a.\sin b.\sin c};$$

$$=\frac{n^2}{\sin s.\sin a.\sin b.\sin c}.$$

$$\cos\tfrac{1}{2}A.\cos\tfrac{1}{2}B.\cos\tfrac{1}{2}C=\frac{n.\sin s}{\sin a.\sin b.\sin c}.$$

$$\tan\tfrac{1}{2}A.\tan\tfrac{1}{2}B.\tan\tfrac{1}{2}C=\frac{n}{(\sin s)^2}.$$

$$\cos\tfrac{1}{2}a.\cos\tfrac{1}{2}b.\cos\tfrac{1}{2}c=\frac{\cos(S-A).\cos(S-B).\cos(S-C)}{\sin A.\sin B.\sin C}$$

$$=-\frac{N^2}{\cos S.\sin A.\sin B.\sin C}.$$

$$\sin\tfrac{1}{2}a.\sin\tfrac{1}{2}b.\sin\tfrac{1}{2}c=\frac{N\cos S}{\sin A.\sin B.\sin C}.$$

$$\cot\tfrac{1}{2}a.\cot\tfrac{1}{2}b.\cot\tfrac{1}{2}c=\frac{N}{(\cos S)^2}.$$

$$n = \tfrac{1}{2} . \left. \overline{(\sin a . \sin b . \sin c)^2 . \sin A . \sin B . \sin C} \right|^{\frac{1}{3}} .$$

$$N = \tfrac{1}{2} . \left. \overline{\sin a . \sin b . \sin c . (\sin A . \sin B . \sin C)^2} \right|^{\frac{1}{3}} .$$

$$\sin s = 2 \frac{n}{N} \cos \tfrac{1}{2} A . \cos \tfrac{1}{2} B . \cos \tfrac{1}{2} C .$$

$$\cos S = 2 \frac{N}{n} \sin \tfrac{1}{2} a . \sin \tfrac{1}{2} b . \sin \tfrac{1}{2} c .$$

$$\cos s = \frac{1 - (\sin \frac{1}{2} A)^2 - (\sin \frac{1}{2} B)^2 - (\sin^2 \frac{1}{2} C)^2}{2 \sin \frac{1}{2} A . \sin \frac{1}{2} B . \sin \frac{1}{2} C} ;$$

$$= \frac{1 - \cos A - \cos B - \cos C}{4 \sin \frac{1}{2} A . \sin \frac{1}{2} B . \sin \frac{1}{2} C} .$$

$$\sin S = \frac{1 - (\cos \frac{1}{2} a)^2 - (\cos \frac{1}{2} b)^2 - (\cos \frac{1}{2} c)^2}{2 \cos \frac{1}{2} a . \cos \frac{1}{2} b . \cos \frac{1}{2} c} ;$$

$$= \frac{1 + \cos a + \cos b + \cos c}{4 \cos \frac{1}{2} a . \cos \frac{1}{2} b . \cos \frac{1}{2} c} .$$

(*L*. 181—201, tab. VIII.)

(23.) *Formulæ for the area* (*E*) *of a spherical triangle.*

$$E = \pi r^2 . \left(\frac{S}{90^\circ} - 1 \right) ;$$

$= 2S - \pi$, the radius being the linear unit;

$= 57^\circ{,}2957795\,(2S - 180^\circ)$, the value in *square degrees.*

$$\sin \tfrac{1}{2} E = \frac{n}{2 \cos \frac{1}{2} a . \cos \frac{1}{2} b . \cos \frac{1}{2} c} .$$

$$\cos \tfrac{1}{2} E = \frac{\cos \frac{1}{2} a . \cos \frac{1}{2} b + \sin \frac{1}{2} a . \sin \frac{1}{2} b . \cos C}{\cos \frac{1}{2} c} ;$$

$$= \frac{\cos a + \cos b + \cos c + 1}{4 \cos \frac{1}{2} a . \cos \frac{1}{2} b . \cos \frac{1}{2} c} ;$$

$$= \frac{(\cos \frac{1}{2} a)^2 + (\cos \frac{1}{2} b)^2 + (\cos \frac{1}{2} c)^2 - 1}{2 \cos \frac{1}{2} a . \cos \frac{1}{2} b . \cos \frac{1}{2} c} .$$

$$\tan \tfrac{1}{4} E = \left. \overline{\tan \tfrac{1}{2} s . \tan \tfrac{1}{2} (s - a) . \tan \tfrac{1}{2} (s - b) . \tan \tfrac{1}{2} (s - c)} \right|^{\frac{1}{2}} .$$

$$\tan\tfrac{1}{2}E = \frac{\tan\frac{1}{2}a.\tan\frac{1}{2}b.\sin C}{1+\tan\frac{1}{2}a.\tan\frac{1}{2}b.\cos C}.$$

$$= \frac{\sin\frac{1}{2}a.\sin\frac{1}{2}b.\sin C}{4\cos\frac{1}{2}(\phi+\theta).\cos\frac{1}{2}(\phi-\theta)};$$

$$\text{if } \cos\theta = \sin\tfrac{1}{2}a.\sin\tfrac{1}{2}b.(\cos\tfrac{1}{2}C)^2,$$

$$\text{and } \cos\phi = \tfrac{1}{2}\cos\tfrac{1}{2}(a+b).$$

(*W.* Ch. viii. prop. 13; *Leg. Geom.* Note x; *L.* 177, 263—6.)

(24.) Let R_1, R_2 be the circular radii of the inscribed, and circumscribing circles,

$$\tan R_1 = \frac{n}{\sin s} = \frac{N}{2\cos\frac{1}{2}A.\cos\frac{1}{2}B.\cos\frac{1}{2}C}.$$

$$\tan R_2 = \frac{-\cos s}{N} = \frac{2\sin\frac{1}{2}a.\sin\frac{1}{2}b.\sin\frac{1}{2}c}{n}. \qquad (L.\ 270,\ 1.)$$

(25.) *Solution of right-angled spherical triangles.*

Let A, B, be the oblique angles, C, the right angle;
a, b, the sides, c, the hypothenuse.

Naper's Rules. The *circular* parts are

$$90°-A,\quad 90°-B,\quad a,\quad b,\quad 90°-c;$$

any one of which being called the *middle* part, M,
the two *adjacent* to M on each side of it, A_1, A_2,
and the two remaining or *opposite* parts, O_1, O_2;

$$\sin M = \tan A_1.\tan A_2,$$
$$= \cos O_1.\cos O_2.$$

[1] $\cos c = \cot A.\cot B$;

[2] $\cos c = \cos a.\cos b$;

[3] $\sin a = \tan b.\cot B$;

[4] $\sin a = \sin c.\sin A$;

[5] $\cos A = \tan b.\cot c$;

[6] $\cos A = \cos a.\sin B$.

These are all the forms *essentially* different; four more analagous to [3], [4], [5], [6], may be obtained by changing A, a, into B, b, and *vice versâ*.

These rules may be applied to a quadrantal triangle, in which $c = 90°$, if

$$A, \quad B, \quad -(90° - C), \quad 90° - a, \quad 90° - b,$$

be taken as the circular parts.

Any angle, and the side opposite are either both $> 90°$, or both $< 90°$.

An oblique angle cannot be less, if acute, nor greater, if obtuse, than the opposite side.

The sides are both $> 90°$, or both $< 90°$, if the hypothenuse $< 90°$; and one side $> 90°$, and the other $< 90°$, if the hypothenuse $> 90°$.

A side is $>$ or $<$ hypothenuse, according as it is $>$ or $< 90°$.

$A + B$ and $a + b$ are both $>$, or both $<$, 180°.

$A + B$ is always $> 90°$, and $A \sim B$ always $< 90°$.

(*W.* Chap. x; *L.* 202—9.)

[1] Given A, c: $\sin a = \sin c . \sin A$;

$\log \text{Sin}\, a = \log \text{Sin}\, c + \log \text{Sin}\, A - \log r.$

$$\tan b = \tan c . \cos A;$$

$\log \text{Tan}\, b = \log \text{Tan}\, c + \log \text{Cos}\, A - \log r.$

$$\cot B = \cos c . \tan A;$$

$\log \text{Cot}\, B = \log \text{Cos}\, c + \log \text{Tan}\, A - \log r.$

[2] Given a, c: $\sin A = \dfrac{\sin a}{\sin c}$;

$\log \text{Sin}\, A = \log \text{Sin}\, a - \log \text{Sin}\, c + \log r.$

$$\cos b = \frac{\cos c}{\cos a};$$

$\log \text{Cos}\, b = \log \text{Cos}\, c - \log \text{Cos}\, a + \log r.$

$$\cos B = \tan a . \cot c;$$

$\log \text{Cos}\, B = \log \text{Tan}\, a + \log \text{Cot}\, c - \log r.$

[3] Given a, b: $\cos c = \cos a . \cos b$;

$\log \mathrm{Cos}\, c = \log \mathrm{Cos}\, a + \log \mathrm{Cos}\, b - \log r.$

$$\tan A = \frac{\tan a}{\sin b};$$

$\log \mathrm{Tan}\, A = \log \mathrm{Tan}\, a - \log \mathrm{Sin}\, b + \log r.$

[4] Given A, a: $\sin c = \dfrac{\sin a}{\sin A}$;

$\log \mathrm{Sin}\, c = \log \mathrm{Sin}\, a - \log \mathrm{Sin}\, A + \log r.$

$$\sin b = \tan a . \cot A;$$

$\log \mathrm{Sin}\, b = \log \mathrm{Tan}\, a + \log \mathrm{Cot}\, A - \log r.$

$$\sin B = \frac{\cos A}{\cos a};$$

$\log \mathrm{Sin}\, B = \log \mathrm{Cos}\, A - \log \mathrm{Cos}\, a + \log r.$

[5] Given A, b: $\tan c = \dfrac{\tan b}{\cos A}$;

$\log \mathrm{Tan}\, c = \log \mathrm{Tan}\, b - \log \mathrm{Cos}\, A + \log r.$

$$\tan a = \sin b . \tan A;$$

$\log \mathrm{Tan}\, a = \log \mathrm{Sin}\, b + \log \mathrm{Tan}\, A - \log r.$

$$\cos B = \sin A . \cos b;$$

$\log \mathrm{Cos}\, B = \log \mathrm{Sin}\, A + \log \mathrm{Cos}\, b - \log r.$

[6] Given A, B: $\cos c = \cot A . \cot B$;

$\log \mathrm{Cos}\, c = \log \mathrm{Cot}\, A + \log \mathrm{Cot}\, B - \log r.$

$$\cos a = \frac{\cos A}{\cos B},$$

$\log \mathrm{Cos}\, a = \log \mathrm{Cos}\, A - \log \mathrm{Cos}\, B + \log r.$

The results obtained in [2], [4], are doubtful; the ambiguity may in some cases be removed by attending to the conditions stated in page 118.

When a small angle is to be determined from its cosine, or an angle nearly $=90°$ from its sine, a small error in the sine or cosine gives a large one in the angle; in such cases some of the following formulæ will give more accurate results.

$$\tan\tfrac{1}{2}c=\left.\overline{\frac{-\cos(A+B)}{\cos(A-B)}}\right|^{\frac{1}{2}}.$$

$$\tan\tfrac{1}{2}a=\left.\overline{\frac{\tan(\frac{1}{2}\overline{A+B}-45°)}{\tan(\frac{1}{2}\overline{A-B}-45°)}}\right|^{\frac{1}{2}}.$$

$$\tan\tfrac{1}{2}a=\left.\overline{\tan\tfrac{1}{2}(c+b).\tan\tfrac{1}{2}(c-b)}\right|^{\frac{1}{2}}.$$

$$\tan\tfrac{1}{2}A=\left.\overline{\frac{\sin(c-b)}{\sin(c+b)}}\right|^{\frac{1}{2}}.$$

$$\tan(45°-\tfrac{1}{2}a)=\left.\overline{\tan(45°-x)}\right|^{\frac{1}{2}};\ \text{if } \tan x=\sin c.\sin A.$$

$$\tan(45°+\tfrac{1}{2}A)=\left.\overline{\frac{\tan\frac{1}{2}(c+a)}{\tan\frac{1}{2}(c-a)}}\right|^{\frac{1}{2}}.$$

$$\tan(45°+\tfrac{1}{2}c)=\left.\overline{\frac{\tan\frac{1}{2}(A+a)}{\tan\frac{1}{2}(A-a)}}\right|^{\frac{1}{2}}.$$

$$\tan(45°+\tfrac{1}{2}b)=\left.\overline{\frac{\sin(A+a)}{\sin(A-a)}}\right|^{\frac{1}{2}}.$$

$$\tan(45°+\tfrac{1}{2}B)=\left.\overline{\frac{\cot\frac{1}{2}(A+a)}{\tan\frac{1}{2}(A-a)}}\right|^{\frac{1}{2}}.$$

Formulæ adapted to a table of natural sines.

$\text{Sin}\,a=\frac{1}{2}\text{Cos}\,(c-A)-\frac{1}{2}\text{Cos}\,(c+A).$

$\text{Cos}\,c=\frac{1}{2}\text{Cos}\,(a+b)+\frac{1}{2}\text{Cos}\,(a-b).$

$\text{Cos}\,A=\frac{1}{2}\text{Sin}\,(a+B)-\frac{1}{2}\text{Sin}\,(a-B).$

(*W.* Ch. x; *L.* §. v; *C.* Ch. xvii.)

(26.) *Solution of oblique-angled spherical triangles.*

First Case: given a, b, c.

[1] $\sin A = \dfrac{2}{\sin b . \sin c} . \overline{\sin s . \sin(s-a) . \sin(s-b) . \sin(s-c)}\Big|^{\frac{1}{2}}$;

$\log \text{Sin}\, A = \log 2 + \log r - \log \text{Sin}\, b - \log \text{Sin}\, c$
$+ \frac{1}{2}\{\log \text{Sin}\, s + \log \text{Sin}\,(s-a) + \log \text{Sin}\,(s-b) + \log \text{Sin}\,(s-c)\}$.

[2] $(\sin \frac{1}{2}A)^2 = \dfrac{\sin(s-b) . \sin(s-c)}{\sin b . \sin c}$;

$\log \text{Sin}\, \frac{1}{2}A = \frac{1}{2}\{\log \text{Sin}\,(s-b) + \log \text{Sin}\,(s-c)$
$- \log \text{Sin}\, b - \log \text{Sin}\, c\} + \log r$.

[3] $(\cos \frac{1}{2}A)^2 = \dfrac{\sin s . \sin(s-a)}{\sin b . \sin c}$;

$\log \text{Cos}\, \frac{1}{2}A = \frac{1}{2}\{\log \text{Sin}\, s + \log \text{Sin}\,(s-a)$
$- \log \text{Sin}\, b - \log \text{Sin}\, c\} + \log r$.

[4] $(\tan \frac{1}{2}a)^2 = \dfrac{\sin(s-b) . \sin(s-c)}{\sin s . \sin(s-a)}$;

$\log \text{Tan}\, \frac{1}{2}a = \frac{1}{2}\{\log \text{Sin}\,(s-b) + \log \text{Sin}\,(s-c)$
$- \log \text{Sin}\, s - \log \text{Sin}\,(s-a)\} + \log r$.

[5] Assume $\cos\theta = \cos b . \cos c$,

$\log \text{Cos}\, \theta - \log \text{Cos}\, b + \log \text{Cos}\, c - \log r$; then

$$\cos A = \frac{2 \sin \frac{1}{2}(\theta + a) . \sin \frac{1}{2}(\theta - a)}{\sin b . \sin c};$$

$\log \text{Cos}\, A = \log 2 + \log \text{Sin}\, \frac{1}{2}(\theta + a) + \log \text{Sin}\, \frac{1}{2}(\theta - a)$
$- \log \text{Sin}\, b - \log \text{Sin}\, c + \log r$.

$A - 90°$, and $\theta - a$ have the same sign.

[6] Assume $\tan\phi = \dfrac{\cos b . \cos c}{\sin a}$,

$\log \text{Tan}\, \phi = \log \text{Cos}\, b + \log \text{Cos}\, c - \log \text{Sin}\, a$; then

$$\cos A = \frac{\cos(a+\phi)}{\cos\phi . \sin b . \sin c};$$

$\log \text{Cos}\, A = \log \text{Cos}\,(a+\phi) + 3 \log r$
$- \log \text{Cos}\, \phi - \log \text{Sin}\, b - \log \text{Sin}\, c$.

Second Case: given A, B, C.

$$[1]\ \sin a = \frac{2}{\sin B.\sin C}.\overline{-\cos S.\cos(S-A).\cos(S-B).\cos(S-C)}\Big|^{\frac{1}{2}}.$$

$$\log \operatorname{Sin} a = \log 2 + \log r - \log \operatorname{Sin} B - \log \sin C$$
$$+\tfrac{1}{2}\{\log(-\operatorname{Cos} S) + \log \operatorname{Cos}(S-A) + \log \operatorname{Cos}(S-B)$$
$$+ \log \operatorname{Cos}(S-C)\}.$$

$$[2]\ (\sin \tfrac{1}{2}a)^2 = \frac{-\cos S.\cos(S-A)}{\sin B.\sin C};$$

$$\log \operatorname{Sin} \tfrac{1}{2}a = \log r$$
$$+\tfrac{1}{2}\{\log(-\operatorname{Cos} S) + \log \operatorname{Cos}(S-A) - \log \operatorname{Sin} B - \log \operatorname{Sin} C\}.$$

$$[3]\ (\cos \tfrac{1}{2}a)^2 = \frac{\cos(S-B).\cos(S-C)}{\sin B.\sin C}.$$

$$\log \operatorname{Cos} \tfrac{1}{2}a = \log r$$
$$+\tfrac{1}{2}\{\log \operatorname{Cos}(S-B) + \log \operatorname{Cos}(S-C) - \log \operatorname{Sin} B - \log \operatorname{Sin} C\}.$$

$$[4]\ (\tan \tfrac{1}{2}a)^2 = \frac{-\cos S.\cos(S-A)}{\cos(S-B).\cos(S-C)};$$

$$\log \operatorname{Tan} \tfrac{1}{2}a = \log r + \tfrac{1}{2}\{\log(-\operatorname{Cos} S) + \log \operatorname{Cos}(S-A)$$
$$- \log \operatorname{Cos}(S-B) - \operatorname{Cos}(S-C)\}.$$

[5] Assume $\cos\theta = \cos B.\cos C$;

$\log \operatorname{Cos}\theta = \log \operatorname{Cos} B + \log \operatorname{Cos} C - \log r$: then

$$\cos a = \frac{2\cos\tfrac{1}{2}(A+\theta).\cos\tfrac{1}{2}(A-\theta)}{\sin B.\sin C};$$

$$\log \operatorname{Cos} a = \log r + \log 2 + \log \operatorname{Cos}\tfrac{1}{2}(A+\theta) + \log \operatorname{Cos}\tfrac{1}{2}(A-\theta)$$
$$- \log \operatorname{Sin} B - \log \operatorname{Sin} C.$$

[6] Assume $\tan\phi = \dfrac{\cos B.\cos C}{\sin A}$;

$\log \operatorname{Tan}\phi = \log \operatorname{Cos} B + \log \operatorname{Cos} C - \log \operatorname{Sin} A$: then

$$\cos a = \frac{\cos(A-\phi)}{\sin B.\sin C.\cos\phi};$$

$$\log \operatorname{Cos} a = 3\log r + \log \operatorname{Cos}(A-\phi)$$
$$- \log \operatorname{Sin} B - \log \operatorname{Sin} C - \log \operatorname{Cos}\phi.$$

The same observations as in the analagous cases of plane triangles will apply to the numerical accuracy of the different methods in these two cases.

Third Case: given a, b, C. To determine A and B:

[1] $\tan\frac{1}{2}(A+B) = \dfrac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\frac{1}{2}C$;

$\log \text{Tan}\frac{1}{2}(A+B) = \log \text{Cos}\frac{1}{2}(a-b) + \log \text{Cot}\frac{1}{2}C - \log \text{Cos}\frac{1}{2}(a+b)$.

$$\tan\frac{1}{2}(A-B) = \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}.\cot\frac{1}{2}C;$$

$\log \text{Tan}\frac{1}{2}(A-B) = \log \text{Sin}\frac{1}{2}(a-b) + \log \text{Cot}\frac{1}{2}C - \log \text{Sin}\frac{1}{2}(a+b)$.

$$A = \frac{1}{2}(A+B) + \frac{1}{2}(A-B),$$

$$B = \frac{1}{2}(A+B) - \frac{1}{2}(A-B).$$

[2] Assume $\tan\alpha = \tan a.\cos C$;

$\log \text{Tan}\,\alpha = \log \text{Tan}\,a + \log \text{Cos}\,C - \log r$:

and let $\beta = b - \alpha$, then

$$\tan A = \tan C\,\frac{\sin\alpha}{\sin\beta};$$

$\log \text{Tan}\,A = \log \text{Tan}\,C + \log \text{Sin}\,\alpha - \log \text{Sin}\,\beta$.

To determine c.

[1] $2(\sin\frac{1}{2}c)^2 = \text{vers}(a-b) + \sin a.\sin b.\text{vers}\,C$.

$$\text{Assume } (\tan\theta)^2 = \frac{\sin a.\sin b.\text{vers}\,C}{\text{vers}(a-b)};$$

$\log \text{Tan}\,\theta = \frac{1}{2}\{\log \text{Sin}\,a + \log \text{Sin}\,b + \log \text{Vers}\,C - \log \text{Vers}(a-b)\}$.

then $2(\sin\frac{1}{2}c)^2 = \text{vers}(a-b).(\sec\theta)^2$;

$\log \text{Sin}\frac{1}{2}c = \log \text{Sec}\,\theta + \frac{1}{2}\{\log \text{Vers}(a-b) - \log r - \log 2\}$.

[2] $(\sin\frac{1}{2}c)^2 = (\sin\frac{1}{2}.\overline{a+b})^2 - \sin a.\sin b\,(\cos\frac{1}{2}C)^2$.

Assume $(\sin\theta)^2 = \sin a.\sin b.(\cos\frac{1}{2}C)^2$

$\log \text{Sin}\,\theta = \frac{1}{2}\{\log \text{Sin}\,a + \log \text{Sin}\,b\} + \log \text{Cos}\frac{1}{2}C - \log r$:

then $(\sin\frac{1}{2}c)^2 = \sin(\frac{1}{2}.\overline{a+b}+\theta).\sin(\frac{1}{2}.\overline{a+b}-\theta)$;

$\log\text{Sin}\,\frac{1}{2}c = \frac{1}{2}\{\log\text{Sin}\,(\frac{1}{2}.\overline{a+b}+\theta) + \log\text{Sin}\,(\frac{1}{2}.\overline{a+b}-\theta)\}$.

[3] Assume $\tan\alpha = \cos C.\tan a$;

$\log\text{Tan}\,\alpha = \log\text{Cos}\,C + \log\text{Tan}\,a - \log r$:

and let $\beta = b \sim \alpha$, then

$$\cos c = \cos a.\frac{\cos\beta}{\cos\alpha};$$

$\log\text{Cos}\,c = \log\text{Cos}\,a + \log\text{Cos}\,\beta - \log\text{Cos}\,\alpha$.

If c is *small*, this formula is not sufficiently accurate.

[4] $\sin c = \dfrac{\sin C}{\sin A}\sin a$;

$\log\text{Sin}\,c = \log\text{Sin}\,C + \log\text{Sin}\,a - \log\text{Sin}\,A$.

Fourth Case: given A, B, c. To determine a and b:

[1] $\tan\frac{1}{2}(a+b) = \dfrac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\frac{1}{2}c$;

$\log\text{Tan}\,\frac{1}{2}(a+b) = \log\text{Cos}\,\frac{1}{2}(A-B) + \log\text{Tan}\,\frac{1}{2}c - \log\text{Cos}\,\frac{1}{2}(A-B)$.

$$\tan\frac{1}{2}(a-b) = \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\frac{1}{2}c;$$

$\log\text{Tan}\,\frac{1}{2}(a-b) = \log\text{Sin}\,\frac{1}{2}(A-B) - \log\text{Sin}\,\frac{1}{2}(A+B) + \log\text{Tan}\,\frac{1}{2}c$.

$$a = \tfrac{1}{2}(a+b) + \tfrac{1}{2}(a-b),$$
$$b = \tfrac{1}{2}(a+b) - \tfrac{1}{2}(a-b).$$

[2] Assume $\cot\alpha = \tan A.\cos c$;

$\log\text{Cot}\,\alpha = \log\text{Tan}\,A + \log\text{Cos}\,c - \log r$;

and let $\beta = B \sim \alpha$, then

$$\tan a = \tan c\frac{\cos\alpha}{\cos\beta};$$

$\log\text{Tan}\,a = \log\text{Tan}\,c + \log\text{Cos}\,\alpha - \log\text{Cos}\,\beta$.

Required C:

[1] $\sin C = \sin A \dfrac{\sin c}{\sin a}$;

$\log \text{Sin}\, C = \log \text{Sin}\, A + \log \text{Sin}\, c - \log \text{Sin}\, a$.

[2] $2 (\sin \tfrac{1}{2} C)^2 = 2(\cos \tfrac{1}{2}.\overline{A + B})^2 + \text{vers}\, c . \sin A . \sin B$.

Assume $(\tan \theta)^2 = \dfrac{\text{vers}\, c . \sin A . \sin B}{2 (\cos \frac{1}{2}.\overline{A + B})^2}$;

$\log \text{Tan}\, \theta = \tfrac{1}{2} \{\log \text{Vers}\, c + \log \text{Sin}\, A + \log \text{Sin}\, B + \log r - \log 2\}$
$- \log \text{Cos}\, \tfrac{1}{2}(A + B)$:

then $\sin \tfrac{1}{2} C = \cos \tfrac{1}{2} (A + B) . \sec \theta$;

$\log \text{Sin}\, \tfrac{1}{2} C = \log \text{Cos}\, \tfrac{1}{2} (A + B) + \log \text{Sec}\, \theta - \log r$.

[3] Assume $\cot \alpha = \tan A . \cos c$;

$\log \text{Cot}\, \alpha = \log \text{Tan}\, A + \log \text{Cos}\, c - \log r$:

and let $\beta = B - \alpha$, then

$\cos C = \cos A \dfrac{\sin \beta}{\sin \alpha}$;

$\log \text{Cos}\, C = \log \text{Cos}\, A + \log \text{Sin}\, \beta - \log \text{Sin}\, \alpha$.

This is not sufficiently accurate, if C is small.

Fifth Case: given a, b, A. To determine B:

$$\sin B = \sin A \frac{\sin b}{\sin a};$$

$\log \text{Sin}\, B = \log \text{Sin}\, A + \log \text{Sin}\, b - \log \text{Sin}\, a$.

If $\dfrac{\sin a}{\sin b} < \sin A$, there is no solution.

If $\dfrac{\sin a}{\sin b} = \sin A$, $B = 90°$, when A and a are of the same species; when A and a are of different species, there is no solution.

If $\dfrac{\sin a}{\sin b} > \sin A$, and $< s$, there are two supplemental values of B, when A and a are of the same species; when of different species, there is no solution.

If $\frac{\sin a}{\sin b} \nless 1$, that value only of B is admissible, which is of the same species as b.

To determine C.

[1] $\cot \frac{1}{2} C = \tan \frac{1}{2} (A + B) \frac{\cos \frac{1}{2} (a + b)}{\cos \frac{1}{2} (a - b)}$;

$\log \operatorname{Cot} \frac{1}{2} C = \log \operatorname{Tan} \frac{1}{2} (A + B) + \log \operatorname{Cos} \frac{1}{2} (a + b) - \log \operatorname{Cos} \frac{1}{2} (a - b)$

[2] Assume $\cot \alpha = \tan A . \cos b$;

$\log \operatorname{Cot} \alpha = \log \operatorname{Tan} A + \log \operatorname{Cos} b - \log r$:

and let $\cos \beta = \frac{\cos \alpha . \tan b}{\tan a}$;

$\log \operatorname{Cos} \beta = \log \operatorname{Cos} \alpha + \log \operatorname{Tan} b - \log \operatorname{Tan} a$:

then $C = \alpha \pm \beta$; + or −, according as α and β are of the same or different species, if $\frac{\sin a}{\sin b} > 1$: if $\frac{\sin a}{\sin b} < 1$, and $> \sin A$, then C has two values, $\alpha \pm \beta$.

[3] Assume $\tan \theta = \cos b . \tan A$;

$\log \operatorname{Tan} \theta = \log \operatorname{Cos} b + \log \operatorname{Tan} A - \log r$:

then $\sin (C + \theta) = \frac{\tan b}{\tan a} \sin \theta$;

$\log \operatorname{Sin} (C + \theta) = \log \operatorname{Tan} b + \log \operatorname{Sin} \theta - \log \operatorname{Tan} a$.

To determine c.

[1] $\sin c = \sin a \frac{\sin C}{\sin A}$;

$\log \operatorname{Sin} c = \log \operatorname{Sin} a + \log \operatorname{Sin} C - \log \operatorname{Sin} A$.

[2] $\tan \frac{1}{2} c = \tan \frac{1}{2} (a + b) \frac{\cos \frac{1}{2} (A + B)}{\cos \frac{1}{2} (A - B)}$;

$\log \operatorname{Tan} \frac{1}{2} c = \log \operatorname{Tan} \frac{1}{2} (a + b) + \log \operatorname{Cos} \frac{1}{2} (A + B) - \log \operatorname{Cos} \frac{1}{2} (A - B)$.

[3] Assume $\tan \alpha = \cos A . \tan b$;

$\log \text{Tan}\, \alpha = \log \text{Cos}\, A + \log \text{Tan}\, b - \log r$:

and let $\cos \beta = \dfrac{\cos \alpha . \cos a}{\cos b}$;

$\log \text{Cos}\, \beta = \log \text{Cos}\, \alpha + \log \text{Cos}\, a - \log \text{Cos}\, b$:

then $c = \alpha \pm \beta$, under the same conditions as above.

Sixth Case: given A, B, a. To determine b:

$$\sin b = \sin a \frac{\sin B}{\sin A};$$

$\log \text{Sin}\, b = \log \text{Sin}\, a + \log \text{Sin}\, B - \log \text{Sin}\, A$.

The above remarks on $\dfrac{\sin a}{\sin b}$ may here be applied to $\dfrac{\sin A}{\sin B}$.

C, and c may be determined as above; or thus,

Assume $\cot \alpha = \tan B . \cos a$;

$\log \text{Cot}\, \alpha = \log \text{Tan}\, B + \log \text{Cos}\, a - \log r$:

and let $\sin \beta = \dfrac{\sin \alpha . \cos A}{\cos B}$;

$\log \text{Sin}\, \beta = \log \text{Sin}\, \alpha + \log \text{Cos}\, A - \log \text{Cos}\, B$:

then $C = \alpha \pm \beta$, as above.

Assume $\tan \alpha = \tan a . \cos B$,

$\log \text{Tan}\, \alpha = \log \text{Tan}\, a + \log \text{Cos}\, B - \log r$.

$$\sin \beta = \sin \alpha \frac{\cos A}{\cos B},$$

$\log \text{Sin}\, \beta = \log \text{Sin}\, \alpha + \log \text{Cos}\, A - \log \text{Cos}\, B$;

then $c = \alpha \pm \beta$, as above.

(*W*. Ch. xi; *C*. Ch. xviii; *L*. 221—62: *Legendre, Trig.* 84–91; *Delambre, Astron.* 138—83.)

(27.) *Series for the sine, cosine, &c. in terms of the arc.*

$$\sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1....7} + \&\text{c.}$$ [α]

$$\cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1....6} + \&\text{c.}$$ [β]

$$\sec x = 1 + \frac{x^2}{1.2} + \frac{5x^4}{1.2.3.4} + \frac{61x^6}{1....6} + \frac{5.277x^8}{1.....8} + \frac{50521x^{10}}{1.....10}$$
$$+ \frac{5.540553x^{12}}{1.....12} + \&\text{c.}$$ [γ]

$$\operatorname{cosec} x = \frac{1}{x} + \frac{x}{1.2.3} + \frac{7x^3}{3.1....5} + \frac{31x^5}{3.1....7} + \frac{3.127x^7}{5.1....9}$$
$$+ \frac{5.7.73x^9}{3.1....11} + \frac{1414477x^{11}}{3.5.7.1....13} + \&\text{c.}$$ [δ]

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{3.5} + \frac{17x^7}{3^2.5.7} + \frac{62x^9}{3^4.5.7} + \frac{1382x^{11}}{3^4.5^2.7.11}$$
$$+ \frac{21844x^{13}}{3^5.5^2.7.11.13} + \frac{929569x^{15}}{3^6.5^3.7^2.11.13} + \&\text{c.}$$ [ε]

[α] $\sin x = \overset{\infty}{\mathrm{S}}_m (-1)^{m-1} \dfrac{x^{2m-1}}{\underline{|2m-1}}.$

[β] $\cos x = \overset{\infty}{\mathrm{S}}_m (-1)^{m-1} \dfrac{x^{2m-2}}{\underline{|2m-2}}.$

[γ] $\sec x = \overset{\infty}{\mathrm{S}}_m (-1)^{m-1} x^{2m-2} . \varpi^{m-1} a^{-1}, \left\{ a_{m-1} = \dfrac{1}{\underline{|2m-2}} \right\};$

where $\varpi^{m-1} a^{-1} = \overset{m}{\mathrm{S}}_n (-1)^{n-1} a^{-n} . \varpi^{m-n} a_1{}^{n-1}.$

[δ] $\operatorname{cosec} x = \overset{\infty}{\mathrm{S}}_m (-1)^{m-1} x^{2m-3} . \varpi^{m-1} a^{-1}, \left\{ a_{m-1} = \dfrac{1}{\underline{|2m-1}} \right\}.$

[ε] $\tan x = \overset{\infty}{\mathrm{S}}_m (-1)^{m-1} x^{2m-1} . \overset{m}{\mathrm{S}}_n \dfrac{\varpi^{n-1} a^{-1}}{\underline{|2m-2n+1}}, \left\{ a_{m-1} = \dfrac{1}{\underline{|2m-2}} \right\}.$

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{3^2.5} - \frac{2x^5}{3^3.5.7} - \frac{x^7}{3^3.5^2.7} - \frac{2x^9}{3^5.5.7.11}$$
$$- \frac{1382x^{11}}{3^6.5^3.7^2.11.13} - \frac{4x^{13}}{3^6.5^2.7.11.13} - \frac{3617x^{15}}{3^7.5^4.7^2.11.13.17} - \&c. \quad [\alpha]$$

$$\text{vers}\, x = \frac{x^2}{1.2} - \frac{x^4}{1.2.3.4} + \frac{x^6}{1 \ldots 6} - \&c. \quad [\beta] \qquad (C.\ 266\text{—}302.)$$

(28.) *Series expressing the inverse circular functions.*

$$\sin^{-1}x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7} + \&c. \quad [\gamma]$$

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \&c. \quad [\delta]$$

$$\text{cosec}^{-1}x = \frac{1}{x} + \frac{1}{2}\cdot\frac{1}{3x^3} + \frac{1.3}{2.4}\cdot\frac{1}{5x^5} + \frac{1.3.5}{2.4.6}\cdot\frac{1}{7x^7} + \&c. \quad [\epsilon]$$

$$\text{vers}^{-1}x = (2x)^{\frac{1}{2}}.\left\{1 + \frac{1}{1}\cdot\frac{x}{3.4} + \frac{1.3}{1.2}\cdot\frac{x^2}{5.4^2} + \frac{1.3.5}{1.2.3}\cdot\frac{x^3}{7.4^3} + \&c.\right\} \quad [\theta]$$

$$\cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x. \qquad \cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x.$$

$$\sec^{-1}x = \frac{\pi}{2} - \text{cosec}^{-1}x.$$

[α] $\cot x = \overset{\infty}{\mathrm{S}}_m (-1)^{m-1} x^{2m-3} . \overset{m}{\mathrm{S}}_n \frac{\varpi^{n-1} a^{-1}}{\underline{|2m-2n}}, \ \left\{a_{m-1} = \frac{1}{\underline{|2m-1}}\right\}.$

[β] $\text{vers}\, x = \overset{\infty}{\mathrm{S}}_m (-1)^{m-1} \frac{x^{2m}}{\underline{|2m}}.$

[γ] $\sin^{-1}x = \overset{\infty}{\mathrm{S}}_m \frac{\underline{|1}_{m-1,2}}{\underline{|2}_{m-1,2}} \cdot \frac{x^{2m-1}}{2m-1} = \overset{\infty}{\mathrm{S}}_m \frac{\underline{|1}_{m-1,2}}{\underline{|m-1}} \cdot \frac{x^{2m-1}}{2^{m-1}(2m-1)}.$

[δ] $\tan^{-1}x = \overset{\infty}{\mathrm{S}}_m (-1)^{m-1} \frac{x^{2m-1}}{2m-1}.$

[ε] $\text{cosec}^{-1}x = \overset{\infty}{\mathrm{S}}_m \frac{\underline{|1}_{m-1,2}}{\underline{|m-1}} \cdot \frac{x^{-(2m-1)}}{(2m-1)2^{m-1}}.$

[θ] $\text{vers}^{-1}x = (2x)^{\frac{1}{2}}.\left\{1 + \overset{\infty}{\mathrm{S}}_m \frac{\underline{|1}_{m,2}}{(2m-1).\underline{|m}} \cdot \left(\frac{x}{4}\right)^m\right\}$

(29.) *Series for determining the value of* π.

$$\frac{\pi}{4} = 45^\circ = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \&c.$$

$$\frac{\pi}{6} = 30^\circ = \tfrac{1}{3}\sqrt{3}\left(1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \&c.\right).$$

Euler's Method. $\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$,

$$= \frac{1}{2}\left(1 - \frac{1}{3.4} + \frac{1}{5.4^2} - \frac{1}{7.4^3} + \&c.\right)$$
$$+ \frac{1}{3}\left(1 - \frac{1}{3.9} + \frac{1}{5.9^2} - \frac{1}{7.9^3} + \&c.\right).$$

Machin's Method. $\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$,

$$= \frac{4}{5}\left(1 - \frac{4}{3.100} + \frac{4^2}{5.100^2} - \frac{4^3}{7.100^3} + \&c.\right)$$
$$- \frac{1}{239}\left(1 - \frac{1}{3.57121} + \frac{1}{5.57121^2} - \&c.\right).$$

$\pi = 3{,}14159$ 26535 89793 23846 26433 83279 50288 41971 69399 37510 58209 74944 59230 78164 06286 20899 86280 34825 34211 70679 82148 08651 32823 06647 0938446, &c.

(*Lacroix, Introd. Diff. Calc.* 43—5.)

(30.) *Formulæ involving impossible quantities.*

$$\sin x = \frac{1}{2\sqrt{-1}}\left(\epsilon^{x\sqrt{-1}} - \epsilon^{-x\sqrt{-1}}\right).$$

$$\cos x = \tfrac{1}{2}\left(\epsilon^{x\sqrt{-1}} + \epsilon^{-x\sqrt{-1}}\right).$$

$$\tan x = \frac{1}{\sqrt{-1}} \cdot \frac{\epsilon^{2x\sqrt{-1}} - 1}{\epsilon^{2x\sqrt{-1}} + 1}.$$

$$\epsilon^{\pm x\sqrt{-1}} = \cos x \pm \sqrt{-1}\sin x.$$

$$\cos nx \pm \sqrt{-1}\sin nx = (\cos x \pm \sqrt{-1}\sin x)^n.$$

$$\cos\frac{m}{n}x \pm \sqrt{-1}\sin\frac{m}{n}x = (\cos x \pm \sqrt{-1}\sin x)^{\frac{m}{n}}.$$

$$\sin nx = \frac{1}{2\sqrt{-1}}\left\{(\cos x + \sqrt{-1}\sin x)^n - (\cos x - \sqrt{-1}\sin x)^n\right\}$$

$$\cos nx = \tfrac{1}{2}\left\{(\cos x + \sqrt{-1}\sin x)^n + (\cos x - \sqrt{-1}\sin x)^n\right\}.$$

If $(a+b\sqrt{-1})^{g+h\sqrt{-1}} = \text{Cos}\, c + \sqrt{-1}\, \text{Sin}\, c$,

and $\dfrac{b}{a} = \tan x$, then $c = gx + \frac{1}{2}h . \log(a^2+b^2)$,

$$r = (a^2+b^2)^{\frac{1}{2}g} . \epsilon^{-hx}.$$

$$c = \frac{1}{2\sqrt{-1}} . \log \frac{1+\sqrt{-1}\tan c}{1-\sqrt{-1}\tan c}.$$

(*C.* ch. viii; *Hind*, 247; *L.* 357—62.)

$\log_\epsilon 1 = 2(m-1)\pi\sqrt{-1}$; m being any integer.

Of these values only one is possible, which is obtained by putting $m=1$. (*Lacr. Introd.* 81.)

(31.) *Formulæ for the sums of arcs, and multiple arcs.*

Let the sum of the continued products of each combination of the sines of m of the arcs, a_1, a_2, &c. a_n, into the cosines of the remaining $n-m$ arcs, be represented by $C_{n-m}S_m$; then

$$\sin(a_1+a_2+\&\text{c.}+a_n) = C_n - C_{n-2}S_2 + C_{n-4}S_4 - \&\text{c.}$$

$$\cos(a_1+a_2+\&\text{c.}+a_n) = C_{n-1}S_1 - C_{n-3}S_3 + \&\text{c.}$$

Let T_m be the sum of the continued products of each combination of m tangents,

then $\tan(a_1+a_2+\&\text{c.}+a_n) = \dfrac{T_1 - T_3 + T_5 - \&\text{c.}}{1 - T_2 + T_4 - \&\text{c.}}$.

If $a_1 = a_2 = \&\text{c.} = a_n$, then

$$\sin na = n(\cos a)^{n-1}\sin a - \frac{n(n-1)(n-2)}{1 . 2 . 3}(\cos a)^{n-3}(\sin a)^3 + \&\text{c.}$$

to $\frac{1}{2}(n+1)$ terms, if n is odd, and to $\frac{1}{2}n$ terms, if n is even.

$$\cos na = (\cos a)^n - \frac{n(n-1)}{1 . 2}(\cos a)^{n-2}(\sin a)^2$$

$$+ \frac{n(n-1)(n-2)(n-3)}{1 . 2 . 3 . 4}(\cos a)^{n-4}(\sin a)^4 - \&\text{c.}$$

to $\frac{1}{2}(n+1)$ terms, if n is odd, and to $\frac{1}{2}n+1$ terms, if n is even.

(32.) *Sines and cosines of multiple Arcs.*

[1] Ascending series.

$$\sin 2x = \cos x \,.\, 2 \sin x.$$

$$\sin 4x = \cos x \,.\, \{4 \sin x - 8 (\sin x)^3\}.$$

$$\sin 6x = \cos x \,.\, \{6 \sin x - 32 (\sin x)^3 + 32 (\sin x)^5\}.$$

&c. = &c.

$$\sin mx = \cos x \left\{ m \sin x - \frac{m(m^2 - 2^2)}{1.2.3} (\sin x)^3 + \frac{m(m^2 - 2^2)(m^2 - 4^2)}{1.\ldots.5} (\sin x)^5 - \&c. \right\}.$$

$$\sin 3x = 3 \sin x - 4 (\sin x)^3.$$

$$\sin 5x = 5 \sin x - 20 (\sin x)^3 + 16 (\sin x)^5.$$

$$\sin 7x = 7 \sin x - 56 (\sin x)^3 + 112 (\sin x)^5 - 64 (\sin x)^7.$$

&c. = &c.

$$\sin mx = m \sin x - \frac{m(m^2 - 1^2)}{1.2.3} (\sin x)^3 + \frac{m(m^2 - 1^2)(m^2 - 3^2)}{1.2.3.4.5} (\sin x)^5 - \&c.$$

$$\sin 2x = \sin x \,.\, 2 \cos x.$$

$$\sin 4x = -\sin x \{4 \cos x - 8 (\cos x)^3\}.$$

$$\sin 6x = \sin x \{6 \cos x - 32 (\cos x)^3 + 32 (\cos x)^5\}.$$

&c. = &c.

$$\sin mx = (-1)^{\frac{m-2}{2}} \sin x \left\{ m \cos x - \frac{m(m^2 - 2^2)}{1.2.3} (\cos x)^3 + \frac{m(m^2 - 2^2)(m^2 - 4^2)}{1\ldots5} (\cos x)^5 - \&c. \right\}$$

$$\sin 3x = -\sin x \{1 - 4 (\cos x)^2\}.$$

$$\sin 5x = \sin x \{1 - 12 (\cos x)^2 + 16 (\cos x)^4\}.$$

$$\sin 7x = -\sin x \{1 - 24 (\cos x)^2 + 80 (\cos x)^4 - 64 (\cos x)^6\}.$$

&c. = &c.

$$\begin{aligned}\sin mx = (-1)^{\frac{m-1}{2}} \sin x \Big\{1 &- \frac{m^2-1^2}{1.2}(\cos x)^2 \\ &+ \frac{(m^2-1^2)(m^2-3^2)}{1.2.3.4}(\cos x)^4 - \text{\&c.}\Big\}\end{aligned}$$

$$\cos 2x = -1 + 2(\cos x)^2.$$

$$\cos 4x = +1 - 8(\cos x)^2 + 8(\cos x)^4.$$

$$\cos 6x = -1 + 18(\cos x)^2 - 48(\cos x)^4 + 32(\cos x)^6.$$

$$\text{\&c.} \quad = \quad \text{\&c.}$$

$$\cos mx = (-1)^{\frac{m}{2}}\Big\{1 - \frac{m^2}{1.2}(\cos x)^2 + \frac{m^2(m^2-2^2)}{1.2.3.4}(\cos x)^4 - \text{\&c.}\Big\}$$

$$\cos 3x = -3\cos x + 4(\cos x)^3.$$

$$\cos 5x = +5\cos x - 20(\cos x)^3 + 16(\cos x)^5.$$

$$\cos 7x = -7\cos x + 56(\cos x)^3 - 112(\cos x)^5 + 64(\cos x)^7.$$

$$\text{\&c.} \quad = \quad \text{\&c.}$$

$$\begin{aligned}\cos mx = (-1)^{\frac{m-1}{2}}.m\Big\{\cos x &- \frac{(m^2-1^2)}{1.2.3}(\cos x)^3 \\ &+ \frac{(m^2-1^2)(m^2-3^2)}{1.2.3.4.5}(\cos x)^5 - \text{\&c.}\Big\}\end{aligned}$$

$$\cos 2x = 1 - 2(\sin x)^2.$$

$$\cos 4x = 1 - 8(\sin x)^2 + 8(\sin x)^4.$$

$$\cos 6x = 1 - 18(\sin x)^2 + 48(\sin x)^4 - 32(\sin x)^6.$$

$$\text{\&c.} \quad = \quad \text{\&c.}$$

$$\begin{aligned}\cos mx = 1 &- \frac{m^2}{1.2}(\sin x)^2 + \frac{m^2.(m^2-2^2)}{1.2.3.4}(\sin x)^4 \\ &- \frac{m^2.(m^2-2^2)(m^2-4^2)}{1\ldots.6}(\sin x)^6 + \text{\&c.}\end{aligned}$$

$$\cos 3x = \cos x\,\{1 - 4(\sin x)^2\}.$$

$$\cos 5x = \cos x\,\{1 - 12(\sin x)^2 + 16(\sin x)^4\}.$$

$$\cos 7x = \cos x\,\{1 - 24(\sin x)^2 + 80(\sin x)^4 - 64(\sin x)^6\}.$$

$$\text{\&c.} \quad = \quad \text{\&c.}$$

$$\cos mx = \cos x\Big\{1 - \frac{m^2-1^2}{1.2}(\sin x)^2 + \frac{(m^2-1^2)(m^2-3^2)}{1.2.3.4}(\sin x)^4 - \text{\&c.}\Big\}$$

[2] Descending series.

$$2\cos nx = (2\cos x)^n - n\,(2\cos x)^{n-2} + \frac{n(n-3)}{1.2}(2\cos x)^{n-4} - \frac{n(n-4)(n-5)}{1.2.3}(2\cos x)^{n-6} + \&c.$$

If n is even, the number of terms is $\frac{1}{2}n+1$, and the last term, $2(-1)^{\frac{1}{2}n}$.

If n is odd, the number of terms is $\frac{1}{2}(n+1)$, and the last term, $(-1)^{\frac{1}{2}(n-1)}(2n\cos x)$. [α]

$$\sin nx = \sin x\left\{(2\cos x)^{n-1} - \frac{(n-2)}{1}(2\cos x)^{n-3} + \frac{(n-3)(n-4)}{1\ .\ 2}(2\cos x)^{n-5} - \&c.\right\}$$ [β]

If n is even, the number of terms is $\frac{1}{2}n$, and the last term, $(-1)^{\frac{1}{2}n-1}.n\cos x$. If n is odd, the number of terms is $\frac{1}{2}(n+1$, and the last term, $(-1)^{\frac{1}{2}(n-1)}$.

$$2\cos nx = (-1)^{\frac{1}{2}n}\left\{(2\sin x)^n - n\,(2\sin x)^{n-2} + \frac{n(n-3)}{2}(2\sin x)^{n-4} - \frac{n(n-4)(n-5)}{2.3}(2\sin x)^{n-6} + \&c.\right.$$ [γ]

[α] $$2\cos nx = n\overset{r}{\mathrm{S}}_m(-1)^{m-1}\frac{\underline{\left|n-m\right.}_{\,m-2}}{\underline{\left|m-1\right.}}(2\cos x)^{n-2m+2}.$$

$r=\frac{1}{2}n+1$ if n is even; $r=\frac{1}{2}(n+1)$ if n is odd.

[β] $$\sin nx = \sin x\,.\,\overset{r}{\mathrm{S}}_m(-1)^{m-1}\frac{\underline{\left|n-m\right.}_{\,m-1}}{\underline{\left|m-1\right.}}(2\cos x)^{n-2m+1}.$$

$r=\frac{1}{2}n$ if n is even; $r=\frac{1}{2}(n+1)$ if n is odd.

[γ] If n is even,

$$2\cos nx = n\overset{r}{\mathrm{S}}_m(-1)^{m+\frac{1}{2}n-1}\frac{\underline{\left|n-m\right.}_{\,m-2}}{\underline{\left|m-1\right.}}(2\sin x)^{n-2m-2};$$

$$\sin nx = \sin x\,.\,\overset{r}{\mathrm{S}}_m(-1)^{m+\frac{1}{2}n-1}\frac{\underline{\left|n-m\right.}_{\,m-1}}{\underline{\left|m-1\right.}}(2\sin x)^{n-2m+1}.$$

$$\sin nx = (-1)^{\frac{1}{2}n}.\cos x\left\{(2\sin x)^{n-1} - \frac{(n-2)}{1}(2\sin x)^{n-3}\right.$$

$$\left. + \frac{(n-3)(n-4)}{1\ .\ 2}(2\sin x)^{n-5} - \&c.\right\}$$

$$2\sin nx = (-1)^{\frac{1}{2}(n-1)}\left\{(2\sin x)^{n} - n(2\sin x)^{n-2}\right.$$

$$\left. + \frac{n(n-3)}{2}(2\sin x)^{n-4} - \frac{n(n-4)(n-5)}{2\ .\ 3}(2\sin x)^{n-6} + \&c.\right\}$$

$$\cos nx = (-1)^{\frac{1}{2}(n-1)}.\cos x\left\{(2\sin x)^{n-1} - \frac{n-2}{1}(2\sin x)^{n-3}\right.$$

$$\left. + \frac{(n-3)(n-4)}{1\ .\ 2}(2\sin x)^{n-5} - \&c.\right\}\ [\delta]$$

n is even in the first and second of these series, and odd in the third and fourth: the last term and number of terms may be determined as in the two first series.

These series are true only when n is a positive integer. (*L.* 370–97; *W.* Ch. iii; *C.* 466–84; *Lagr. Calc. des Fonc. Leç.* 11.)

(33.) *Tangents of multiple arcs.*

$$\tan 2x = \frac{2\tan x}{1-(\tan x)^2}.$$

$$\tan 3x = \frac{3\tan x - (\tan x)^3}{1-3(\tan x)^2}.$$

$$\tan 4x = \frac{4\tan x - 4(\tan x)^3}{1-6(\tan x)^2+(\tan x)^4}$$

[δ] If n is odd,

$$2\sin nx = n\overset{r}{\mathrm{S}}_m(-1)^{m+\frac{1}{2}(n+1)}\frac{\underline{\left|n-m\right.}_{\,m-2}}{\underline{\left|m-1\right.}}(2\sin x)^{n-2m-2};$$

$$\cos nx = \cos x\overset{r}{\mathrm{S}}_m(-1)^{m+\frac{1}{2}(n+1)}\frac{\underline{\left|n-m\right.}_{\,m-1}}{\underline{\left|m-1\right.}}(2\sin x)^{n-2m+1}.$$

$$\tan 5x = \frac{5\tan x - 10(\tan x)^3 + (\tan x)^5}{1 - 10(\tan x)^2 + 5(\tan x)^4}.$$

$$\tan nx = \frac{n\tan x - \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}(\tan x)^3 + \frac{n(n-1)\ldots(n-4)}{1\,.\,2\,\ldots\,5}(\tan x)^5 - \&\text{c.}}{1 - \frac{n(n-1)}{1\,.\,2}(\tan x)^2 + \frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}(\tan x)^4 - \&\text{c.}}.\text{[α]}$$

If n is odd the numerator and denominator must each be continued to $\frac{1}{2}(n+1)$ terms; if even, to $\frac{1}{2}n$, and $\frac{1}{2}n+1$ terms respectively. (*C.* 494—9; *W.* Ch. iii.)

(34.) *Powers of the sine and cosine of an arc.*

$2(\sin x)^2 = -(\cos 2x - 1).$

$2^3(\sin x)^4 = \cos 4x - 4\cos 2x + 3.$

$2^5(\sin x)^6 = -(\cos 6x - 6\cos 4x + 15\cos 2x - 10).$

&c. = &c.

$$2^{n-1}(\sin x)^n = (-1)^{\frac{n}{2}}\left\{\cos nx - n\cos(n-2)x + \frac{n(n-1)}{1\,.\,2}\cos(n-4)x - \&\text{c.}\right\} + 2^{\frac{n}{2}-1}\cdot\frac{1.3.5..(n-1)}{1.2.3\ldots\frac{1}{2}n}.$$

$2^2(\sin x)^3 = -(\sin 3x - 3\sin x).$

$2^4(\sin x)^5 = \sin 5x - 5\sin 3x + 10\sin x.$

$2^6(\sin x)^7 = -(\sin 7x - 7\sin 5x + 21\sin 3x - 35\sin x).$

&c. = &c.

$$2^{n-1}(\sin x)^n = (-1)^{\frac{n-1}{2}}\left\{\sin nx - n\sin(n-2)x + \frac{n(n-1)}{1\,.\,2}\sin(n-4)x - \&\text{c.}\right\}$$

[α]
$$\tan nx = \frac{\overset{r}{\text{S}}_m(-1)^{m-1}\frac{\underline{|n}_{\,2m-1}}{\underline{|2m-1}}(\tan x)^{2m-1}}{\overset{s}{\text{S}}_m(-1)^{m-1}\frac{\underline{|n}_{\,2m-2}}{\underline{|2m-2}}(\tan x)^{2m-2}}:$$

If n is odd, $r = s = \frac{1}{2}(x+1)$: if n is even, $r = \frac{1}{2}n$, and $s = \frac{1}{2}n + 1$.

$2\,(\cos x)^2 = \cos 2x + 1.$

$2^2(\cos x)^3 = \cos 3x + 3\cos x.$

$2^3(\cos x)^4 = \cos 4x + 4\cos 2x + 3.$

$2^4(\cos x)^5 = \cos 5x + 5\cos 3x + 10\cos x.$

$2^5(\cos x)^6 = \cos 6x + 6\cos 4x + 15\cos 2x + 10.$

$2^6(\cos x)^7 = \cos 7x + 7\cos 5x + 21\cos 3x + 35\cos x.$

$$2^{n-1}(\cos x)^n = \cos nx + n\cos(n-2)x + \frac{n(n-1)}{1\,.\;2}\cos(n-4)x + \&c.$$

If n is odd, the number of terms is $\frac{1}{2}(n+1)$, and the last term is $\dfrac{1.3.5\dots\quad n}{1.2.3\dots\frac{1}{2}(n+1)}\,2^{\frac{1}{2}(n-1)}\cos x$;

if n is even, the number of terms is $\frac{1}{2}n+1$, and last term is

$$\frac{1.3.5\dots(n-1)}{1.2.3\dots\;\frac{1}{2}n}\,.\,2^{\frac{1}{2}n-1}.$$

The same observations may be applied to the expansion of $(\sin x)^n$: in both these series n must be a positive integer.

(*L.* 400—2; *C.* 503—10.)

SUMS OF TRIGONOMETRIC SERIES.

(35.) $\sin x + \sin(x+a) + \sin(x+2a) + \&c. + \sin(x+\overline{n-1}.a)$

$$= \frac{\sin\frac{1}{2}na}{\sin\frac{1}{2}a}\sin(x+\tfrac{1}{2}\overline{n-1}.a).\quad \beta$$

$\sin x - \sin(x+a) + \sin(x+2a) - \&c. - \sin(x+\overline{2r-1}.a)$

$$= -\frac{\sin ra}{\cos\frac{1}{2}a}\cos(x+\overline{r-\tfrac{1}{2}}.a).\quad \gamma$$

β $\overset{n}{\mathrm{S}}_m \sin(x+\overline{m-1}.a) = \dfrac{\sin\frac{1}{2}na}{\sin\frac{1}{2}a}\sin(x+\tfrac{1}{2}\overline{n-1}.a).$

γ $\overset{2r}{\mathrm{S}}_m(-1)^{m-1}\sin(x+\overline{m-1}.a) = \dfrac{\sin ra}{\cos\frac{1}{2}a}\sin(x+\overline{r-\tfrac{1}{2}}.a).$

$$\cos x + \tfrac{1}{2}\cos 2x + \tfrac{1}{3}\cos 3x + \tfrac{1}{4}\cos 4x + \&c. = -\log_\epsilon (2\sin\tfrac{1}{2}x). \quad {}^{\alpha}$$

(*Lacr. Diff. Calc. Tome* 2, 466.)

If $\tan y = \dfrac{n\sin x}{1+n\cos x}$, then

$$y = n\sin x - \tfrac{1}{2}n^2\sin 2x + \tfrac{1}{3}n^3\sin 3x - \tfrac{1}{4}n^4\sin 4x + \&c. \quad {}^{\beta}$$

(*C.* 513.)

If $\tan y = n\tan x$, then

$$y = x - \frac{1-n}{1+n}\cdot\frac{\sin 2x}{1} + \left(\frac{1-n}{1+n}\right)^2\frac{\sin 4x}{2} - \left(\frac{1-n}{1+n}\right)^3\frac{\sin 6x}{3} + \&c. \quad {}^{\gamma}$$

(*Maddy, Astron.* 62.)

$$\underset{1}{\{}\tfrac{1}{2} - \tfrac{1}{2}\underset{2}{\{}\tfrac{1}{2} + \tfrac{1}{2}\underset{3}{\{}\tfrac{1}{2} + \ldots\ldots + \tfrac{1}{2}\underset{n}{\{}\tfrac{1}{2} + \tfrac{1}{2}\cos x\underset{n}{\}}^{\frac{1}{2}}\ldots\ldots\underset{2}{\}}^{\frac{1}{2}}\underset{1}{\}}^{\frac{1}{2}} = \frac{\sin x}{2^n}$$

$$\underset{1}{\{}\tfrac{1}{2} + \tfrac{1}{2}\underset{2}{\{}\tfrac{1}{2} + \tfrac{1}{2}\underset{3}{\{}\tfrac{1}{2} + \ldots\ldots + \tfrac{1}{2}\underset{n}{\{}\tfrac{1}{2} + \tfrac{1}{2}\cos x\underset{n}{\}}^{\frac{1}{2}}\ldots\ldots\underset{2}{\}}^{\frac{1}{2}}\underset{1}{\}}^{\frac{1}{2}} = \frac{\cos x}{2^n}.$$

(*Hind, Trig.* 81.)

(36.) *Resolution of trigonometrical quantities into factors.*

$$\sin\frac{\pi}{2n}.\sin\frac{3\pi}{2n}.\sin\frac{5\pi}{2n}\ldots\frac{\sin(2n-1)\pi}{2n} = \frac{1}{2^{n-1}} \quad {}^{\delta}$$

$$\sin x.\sec\frac{x}{2}.\sec\frac{x}{2^2}.\sec\frac{x}{2^3}\ldots\sec\frac{x}{2^n} = 2^n\sin\frac{x}{2^n}. \quad {}^{\epsilon}$$

$^{\alpha}$ $\overset{\infty}{\mathrm{S}}_m \dfrac{\cos mx}{m} = -\log_\epsilon (2\sin\tfrac{1}{2}x).$

$^{\beta}$ $y = \overset{\infty}{\mathrm{S}}_m (-1)^{m-1}\dfrac{n^m.\sin mx}{m}.$

$^{\gamma}$ $y = x + \overset{\infty}{\mathrm{S}}_m\left(-\dfrac{1-n}{1+n}\right)^m.\dfrac{\sin 2mx}{m}.$

$^{\delta}$ $\overset{n}{\mathrm{P}}_m \sin\dfrac{(2m-1)\pi}{2n} = \dfrac{1}{2^{n-1}}.$

$^{\epsilon}$ $\sin x.\overset{n}{\mathrm{P}}_m \sec\dfrac{x}{2^m} = 2^n\sin\dfrac{x}{2^n}.$

$$\sin x = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{(2\pi)^2}\right)\left(1-\frac{x^2}{(3\pi)^2}\right)\text{ \&c.}$$ [α]

$$\cos x = \left(1-\frac{4x^2}{\pi^2}\right)\left(1-\frac{4x^2}{(3\pi)^2}\right)\left(1-\frac{4x^2}{(5\pi)^2}\right)\text{ \&c.}$$ [β]

$$\frac{\pi}{2}=\frac{2.2.4.4.6.6.8.8.\text{ \&c.}}{1.3.3.5.5.7.7.9.\text{ \&c.}}.$$ [γ]

$$\epsilon^x+\epsilon^{\pm y}=\epsilon^{\frac{1}{2}(x\pm y)}\left(1-\left.\overline{\frac{x\mp y}{\pi}}\right|^2\right)\left(1-\left.\overline{\frac{x\mp y}{3\pi}}\right|^2\right).\text{ \&c.}$$ [δ]

$$\epsilon^x-\epsilon^{\pm y}=\epsilon^{\frac{1}{2}(x\pm y)}.\tfrac{1}{2}(x\mp y)\left(1-\left.\overline{\frac{x\mp y}{2\pi}}\right|^2\right)\left(1-\left.\overline{\frac{x\mp y}{4\pi}}\right|^2\right)\text{ \&c.}$$ [ε]

(*L.* 420—5: *Lacr. Diff. Calc. Tome* 3, Ch. vi.)

APPROXIMATE SOLUTION OF TRIANGLES.

(37.) Given A, B, c; A and B being small,

$$a+b-c=\tfrac{1}{2}ABc,\text{ nearly.}$$

Given a, b, and the contained angle, $\pi-\theta$; θ being small,

$$c=a+b-\frac{ab\theta^2}{2(a+b)},\text{ nearly;}$$

$$A=\frac{a\theta}{a+b}+\frac{ab(a-b)}{(a+b)^2}.\frac{\theta^3}{6},\text{ nearly.}\qquad (\textit{Leg. } 97\text{–}9.)$$

[α] $$\sin x = x.\overset{\infty}{\mathrm{P}}_m\left(1-\frac{x^2}{(m\pi)^2}\right).$$

[β] $$\cos x=\overset{\infty}{\mathrm{P}}_m\left(1-\left.\overline{\frac{2x}{(2m-1)\pi}}\right|^2\right).$$

[γ] $$\frac{\pi}{2}=\overset{\infty}{\mathrm{P}}_m\frac{(2m)^2}{(2m-1)(2m+1)}.$$

[δ] $$\epsilon^x+\epsilon^{\pm y}=\epsilon^{\frac{1}{2}(x\pm y)}\overset{\infty}{\mathrm{P}}_m\left(1-\left.\overline{\frac{x\pm y}{(2m-1)\pi}}\right|^2\right).$$

[ε] $$\epsilon^x-\epsilon^{\pm y}=\epsilon^{\frac{1}{2}(x\pm y)}.\tfrac{1}{2}(x\mp y).\overset{\infty}{\mathrm{P}}_m\left(1-\left.\overline{\frac{x\mp y}{2m\pi}}\right|^2\right).$$

$$\cos x + \tfrac{1}{2}\cos 2x + \tfrac{1}{3}\cos 3x + \tfrac{1}{4}\cos 4x + \&\text{c.} = -\log_e(2\sin\tfrac{1}{2}x).$$ [α]

(*Lacr. Diff. Calc. Tome* 2, 466.)

If $\tan y = \dfrac{n\sin x}{1+n\cos x}$, then

$$y = n\sin x - \tfrac{1}{2}n^2\sin 2x + \tfrac{1}{3}n^3\sin 3x - \tfrac{1}{4}n^4\sin 4x + \&\text{c.}$$ [β]

(*C.* 513.)

If $\tan y = n\tan x$, then

$$y = x - \frac{1-n}{1+n}\cdot\frac{\sin 2x}{1} + \left(\frac{1-n}{1+n}\right)^2\frac{\sin 4x}{2} - \left(\frac{1-n}{1+n}\right)^3\frac{\sin 6x}{3} + \&\text{c.}$$ [γ]

(*Maddy, Astron.* 62.)

$$\underset{1}{\{}\tfrac{1}{2} - \tfrac{1}{2}\underset{2}{\{}\tfrac{1}{2} + \tfrac{1}{2}\underset{3}{\{}\tfrac{1}{2} + \dots\dots + \tfrac{1}{2}\underset{n}{\{}\tfrac{1}{2} + \tfrac{1}{2}\cos x\underset{n}{\}}^{\frac{1}{2}}\dots\dots\underset{2}{\}}^{\frac{1}{2}}\underset{1}{\}}^{\frac{1}{2}} = \frac{\sin x}{2^n}$$

$$\underset{1}{\{}\tfrac{1}{2} + \tfrac{1}{2}\underset{2}{\{}\tfrac{1}{2} + \tfrac{1}{2}\underset{3}{\{}\tfrac{1}{2} + \dots\dots + \tfrac{1}{2}\underset{n}{\{}\tfrac{1}{2} + \tfrac{1}{2}\cos x\underset{n}{\}}^{\frac{1}{2}}\dots\dots\underset{2}{\}}^{\frac{1}{2}}\underset{1}{\}}^{\frac{1}{2}} = \frac{\cos x}{2^n}.$$

(*Hind, Trig.* 81.)

(36.) *Resolution of trigonometrical quantities into factors.*

$$\sin\frac{\pi}{2n}\cdot\sin\frac{3\pi}{2n}\cdot\sin\frac{5\pi}{2n}\dots\frac{\sin(2n-1)\pi}{2n} = \frac{1}{2^{n-1}}$$ [δ]

$$\sin x\cdot\sec\frac{x}{2}\cdot\sec\frac{x}{2^2}\cdot\sec\frac{x}{2^3}\dots\sec\frac{x}{2^n} = 2^n\sin\frac{x}{2^n}.$$ [ε]

[α] $\overset{\infty}{\text{S}}_m\dfrac{\cos mx}{m} = -\log_e(2\sin\tfrac{1}{2}x).$

[β] $y = \overset{\infty}{\text{S}}_m(-1)^{m-1}\dfrac{n^m.\sin mx}{m}.$

[γ] $y = x + \overset{\infty}{\text{S}}_m\left(-\dfrac{1-n}{1+n}\right)^m\cdot\dfrac{\sin 2mx}{m}.$

[δ] $\overset{n}{\text{P}}_m\sin\dfrac{(2m-1)\pi}{2n} = \dfrac{1}{2^{n-1}}.$

[ε] $\sin x.\overset{n}{\text{P}}_m\sec\dfrac{x}{2^m} = 2^n\sin\dfrac{x}{2^n}.$

$$\sin x = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{(2\pi)^2}\right)\left(1-\frac{x^2}{(3\pi)^2}\right)\ \&\text{c.}\quad {}^{\alpha}$$

$$\cos x = \left(1-\frac{4x^2}{\pi^2}\right)\left(1-\frac{4x^2}{(3\pi)^2}\right)\left(1-\frac{4x^2}{(5\pi)^2}\right)\ \&\text{c.}\quad {}^{\beta}$$

$$\frac{\pi}{2} = \frac{2.2.4.4.6.6.8.8.\ \&\text{c.}}{1.3.3.5.5.7.7.9.\ \&\text{c.}}.\quad {}^{\gamma}$$

$$\epsilon^x + \epsilon^{\pm y} = \epsilon^{\frac{1}{2}(x\pm y)}\left(1-\left.\overline{\frac{x\mp y}{\pi}}\right|^2\right)\left(1-\left.\overline{\frac{x\mp y}{3\pi}}\right|^2\right).\ \&\text{c.}\quad {}^{\delta}$$

$$\epsilon^x - \epsilon^{\pm y} = \epsilon^{\frac{1}{2}(x\pm y)}.\tfrac{1}{2}(x\mp y)\left(1-\left.\overline{\frac{x\mp y}{2\pi}}\right|^2\right)\left(1-\left.\overline{\frac{x\mp y}{4\pi}}\right|^2\right)\ \&\text{c.}\quad {}^{\epsilon}$$

(*L.* 420—5: *Lacr. Diff. Calc. Tome* 3, Ch. vi.)

APPROXIMATE SOLUTION OF TRIANGLES.

(37.) Given A, B, c; A and B being small,

$$a+b-c=\tfrac{1}{2}ABc, \text{ nearly.}$$

Given a, b, and the contained angle, $\pi-\theta$; θ being small,

$$c=a+b-\frac{ab\theta^2}{2(a+b)}, \text{ nearly;}$$

$$A=\frac{a\theta}{a+b}+\frac{ab(a-b)}{(a+b)^2}.\frac{\theta^3}{6}, \text{ nearly.}\qquad (\textit{Leg. } 97\text{–}9.)$$

[α] $\sin x = x.\overset{\infty}{\mathrm{P}}_m\left(1-\dfrac{x^2}{(m\pi)^2}\right).$

[β] $\cos x = \overset{\infty}{\mathrm{P}}_m\left(1-\left.\overline{\dfrac{2x}{(2m-1)\pi}}\right|^2\right).$

[γ] $\dfrac{\pi}{2} = \overset{\infty}{\mathrm{P}}_m\dfrac{(2m)^2}{(2m-1)(2m+1)}.$

[δ] $\epsilon^x + \epsilon^{\pm y} = \epsilon^{\frac{1}{2}(x\pm y)}\overset{\infty}{\mathrm{P}}_m\left(1-\left.\overline{\dfrac{x\pm y}{(2m-1)\pi}}\right|^2\right).$

[ε] $\epsilon^x - \epsilon^{\pm y} = \epsilon^{\frac{1}{2}(x\pm y)}.\tfrac{1}{2}(x\mp y).\overset{\infty}{\mathrm{P}}_m\left(1-\left.\overline{\dfrac{x\mp y}{2m\pi}}\right|^2\right).$

(38.) *Solution of triangles by series.*

[1] Given the sides a, b, and the angle C of a plane triangle:

$B = \frac{b}{a}\sin C + \frac{b^2}{2a^2}\sin 2C + \frac{b^3}{3a^3}\sin 3C + \&c.$ [α]

$\log_\epsilon c = \log_\epsilon a - \frac{b}{a}\cos C - \frac{b^2}{2a^2}\cos 2C - \frac{b^3}{3a^3}\cos 3C - \&c.$ [β]

[2] Given the sides a, b, and the angle C of a spherical triangle:

$\frac{1}{2}(A+B) = 90^\circ - \frac{1}{2}C + \frac{\tan\frac{1}{2}b}{\cot\frac{1}{2}a}\sin C - \left(\frac{\tan\frac{1}{2}b}{\cot\frac{1}{2}a}\right)^2\frac{\sin 2C}{2} + \&c.$ [γ]

$\frac{1}{2}(A-B) = 90^\circ - \frac{1}{2}C - \frac{\tan\frac{1}{2}b}{\tan\frac{1}{2}a}\sin C - \left(\frac{\tan\frac{1}{2}b}{\tan\frac{1}{2}a}\right)^2\frac{\sin 2C}{2} - \&c.$ [δ]

$$\log_\epsilon \sin\tfrac{1}{2}c = \log_\epsilon(\sin\tfrac{1}{2}a.\cos\tfrac{1}{2}b) - \frac{\tan\frac{1}{2}b}{\tan\frac{1}{2}a}\cos C$$
$$- \left(\frac{\tan\frac{1}{2}b}{\tan\frac{1}{2}a}\right)^2\frac{\cos 2C}{2} - \&c.$$ [ε]

$$\log_\epsilon \cos\tfrac{1}{2}c = \log_\epsilon(\cos\tfrac{1}{2}a.\cos\tfrac{1}{2}b) + \tan\tfrac{1}{2}a.\tan\tfrac{1}{2}b.\cos C$$
$$- (\tan\tfrac{1}{2}a.\tan\tfrac{1}{2}b)^2.\frac{\cos 2C}{2} + \&c.$$ [θ]

[α] $B = \overset{\infty}{S}_m\left(\frac{b}{a}\right)^m.\frac{\sin mC}{m}.$

[β] $\log_\epsilon c = \log_\epsilon a - \overset{\infty}{S}_m\left(\frac{b}{a}\right)^m.\frac{\cos mC}{m}.$

[γ] $\frac{1}{2}(A+B) = 90^\circ - \frac{1}{2}C + \overset{\infty}{S}_m(-1)^{m-1}\left(\frac{\tan\frac{1}{2}b}{\cot\frac{1}{2}a}\right)^m.\frac{\sin mC}{m}.$

[δ] $\frac{1}{2}(A-B) = 90^\circ - \frac{1}{2}C - \overset{\infty}{S}_m\left(\frac{\tan\frac{1}{2}b}{\tan\frac{1}{2}a}\right)^m.\frac{\sin mC}{m}.$

[ε] $\log_\epsilon \sin\frac{1}{2}c = \log_\epsilon(\sin\frac{1}{2}a.\cos\frac{1}{2}b) - \overset{\infty}{S}_m\left(\frac{\tan\frac{1}{2}b}{\tan\frac{1}{2}a}\right)^m.\frac{\cos mC}{m}.$

[θ] $\log_\epsilon \cos\frac{1}{2}c = \log_\epsilon(\cos\frac{1}{2}a.\cos\frac{1}{2}b) + \overset{\infty}{S}_m(-1)^{m-1}(\tan\frac{1}{2}a.\tan\frac{1}{2}b)^m\frac{\cos mC}{m}.$

(39.) *Correction for a compound base.* If the base be composed of two straight lines a and b, inclined at an angle $180° - x''$, the correction is

$$-\frac{abx^2}{a+b}0{,}00000000001175.$$

(40.) ***Reduction of an oblique to a horizontal angle.***

Let α, β, be the angles of elevation of two objects;

γ, their angular distance;

C, the horizontal angle: then

[1] $C - \gamma = \frac{1}{4}\{(\alpha+\beta)^2 \tan\frac{1}{2}\gamma - (\alpha-\beta)^2 \cot\frac{1}{2}\gamma\}$.

[2] $(\sin\frac{1}{2}C)^2 = \dfrac{\sin\frac{1}{2}(a+\overline{\alpha-\beta}) . \sin\frac{1}{2}(a-\overline{\alpha-\beta})}{\cos\alpha . \cos\beta}$;

$$\log \text{Sin}\tfrac{1}{2}C = \tfrac{1}{2}\{\log \text{Sin}\tfrac{1}{2}(a+\overline{\alpha-\beta}) + \log \text{Sin}\tfrac{1}{2}(a-\overline{\alpha-\beta}) - \log \text{Cos}\,\alpha - \log \text{Cos}\,\beta\} + \log r.$$

For a single object, $\cos \textit{reduced} \angle = \dfrac{\cos \textit{observed} \angle}{\cos \angle \textit{ of elevation}}$.

(41.) ***Reduction of a spherical to a plane triangle.***

Let C be the spherical angle,

$C - x$ the corresponding plane angle; then

[1] $x = \dfrac{1}{16}\{(a+b)^2 \tan\frac{1}{2}C - (a-b)^2 \cot\frac{1}{2}C\}$.

[2] *Legendre's Theorem:*

$$x = \frac{\text{area of triangle}}{3r^2} = \tfrac{1}{3}(A+B+C-180°).$$

The value of x in seconds is

$$\frac{\text{area of triangle}}{3r^2 \times 0{,}000004848}.$$

If the area be found in feet, and a degree on the Earth's surface $= 365155$ feet, then $\log_{10}$ *divisor* $= 9{,}803894$.

Reduction of a plane to a spherical triangle.

Let γ be the plane angle contained by the sides a, β;

C the corresponding spherical angle; then

$$\cos C = \frac{\cos\gamma - \frac{1}{4}a\beta}{\overline{(1-\frac{1}{4}a^2)(1-\frac{1}{4}\beta^2)}\Big|^{\frac{1}{2}}}.$$

(*L.* 293—313; *C.* ch. xx; *W.* ch. xii; *Enc. Met.* V. I. p. 698,9.)

FORMULÆ FOR THE CONSTRUCTION OF TABLES.

(42.) $\cos 30^\circ = \overline{\tfrac{1}{2}(1+\tfrac{1}{2})}\Big|^{\frac{1}{2}} = c_1;$

$\cos\tfrac{1}{2}30^\circ = \overline{\tfrac{1}{2}(1+c_1)}\Big|^{\frac{1}{2}} = c_2;$

$\cos\tfrac{1}{4}30^\circ = \overline{\tfrac{1}{2}(1+c_2)}\Big|^{\frac{1}{2}} = c_3;$

&c. = &c.

$\sin 15^\circ = \tfrac{1}{2}(1+\tfrac{1}{2})^{\frac{1}{2}} - \tfrac{1}{2}(1-\tfrac{1}{2})^{\frac{1}{2}} = s_1;$

$\sin\tfrac{1}{2}15^\circ = \tfrac{1}{2}(1+s_1)^{\frac{1}{2}} - \tfrac{1}{2}(1-s_1)^{\frac{1}{2}} = s_2;$

$\sin\tfrac{1}{4}15^\circ = \tfrac{1}{2}(1+s_2)^{\frac{1}{2}} - \tfrac{1}{2}(1-s_2)^{\frac{1}{2}} = s_3;$

&c. = &c.

By continuing this process we obtain $\sin\frac{15^\circ}{2^{10}}$, and thence $\sin 1'$ by taking it proportional to the arc.

$$\frac{15^\circ}{2^{10}} = \left(4'' - \frac{1''}{4}\right)^3 = 52''\,44'''\,3''''\,45^{\text{v}}.$$

Another Method.

$\sin\tfrac{1}{2}a = \overline{\tfrac{1}{2} - \tfrac{1}{2}\{1-(\sin a)^2\}^{\frac{1}{2}}}\Big|^{\frac{1}{2}};$

$\sin\tfrac{1}{4}a = \overline{\tfrac{1}{2} - \tfrac{1}{2}\{1-(\sin\tfrac{1}{2}a)^2\}^{\frac{1}{2}}}\Big|^{\frac{1}{2}};$

&c. = &c.

$$\sin\frac{1}{2^n}a = \overline{\tfrac{1}{2} - \tfrac{1}{2}\left\{1-\left(\sin\frac{1}{2^{n-1}}a\right)^2\right\}^{\frac{1}{2}}}\Bigg|^{\frac{1}{2}}.$$

Let $2\sin 6^\circ = x$, then

$$x^5 - 5x^3 + 5x - 1 = 0,$$

which equation may be solved by approximation.

$$\sin(a+b) = \sin a + \{\sin a - \sin(a-b)\} - 4\sin a.(\sin\tfrac{1}{2}b)^2:$$

Putting $b = 1^\circ$, we obtain Delambre's formula,

$$\sin(a+1^\circ) = \sin a + \{\sin a - \sin(a-1^\circ)\} - 4\sin a.(\sin 30')^2.$$

$$\sin(60^\circ + a) = \sin a + \sin(60^\circ - a).$$

$$\tan(45^\circ + a) = 2\tan 2a + \tan(45^\circ - a).$$

$$\operatorname{cosec} a = \cot a + \tan\tfrac{1}{2}a. \qquad (W. \text{ ch. iv}; \ C. \text{ ch. vi.})$$

(43.) *Sines of arcs expressed by surds.*

$$\sin 3^\circ = \frac{\sqrt{3}+1}{8\sqrt{2}}(\sqrt{5}-1) - \frac{\sqrt{3}-1}{8}\sqrt{5+\sqrt{5}}.$$

$$\sin 6^\circ = -\tfrac{1}{8}(\sqrt{5}+1) + \frac{\sqrt{3}}{4\sqrt{2}}\sqrt{5-\sqrt{5}}.$$

$$\sin 9^\circ = \frac{1}{4\sqrt{2}}(\sqrt{5}+1) - \tfrac{1}{4}\sqrt{5-\sqrt{5}}.$$

$$\sin 12^\circ = -\frac{\sqrt{3}}{8}(\sqrt{5}-1) + \frac{1}{4\sqrt{2}}\sqrt{5+\sqrt{5}}.$$

$$\sin 15^\circ = \frac{1}{2\sqrt{2}}(\sqrt{3}-1).$$

$$\sin 18^\circ = \tfrac{1}{4}(\sqrt{5}-1).$$

$$\sin 21^\circ = \frac{\sqrt{3}-1}{8\sqrt{2}}(\sqrt{5}+1) + \frac{\sqrt{3}+1}{8}\sqrt{5-\sqrt{5}}.$$

$$\sin 24^\circ = \frac{\sqrt{3}}{8}(\sqrt{5}+1) - \frac{1}{4\sqrt{2}}\sqrt{5-\sqrt{5}}.$$

$$\sin 27^\circ = -\frac{1}{4\sqrt{2}}(\sqrt{5}-1) + \tfrac{1}{4}\sqrt{5+\sqrt{5}}.$$

$$\sin 30^\circ = \tfrac{1}{2}.$$

$$\sin 33^\circ = \frac{\sqrt{3}+1}{8\sqrt{2}}(\sqrt{5}-1) + \frac{\sqrt{3}-1}{8}\sqrt{5+\sqrt{5}}.$$

$$\sin 36^\circ = \frac{1}{2\sqrt{2}}\sqrt{5-\sqrt{5}}.$$

$$\sin 39^\circ = \frac{\sqrt{3}+1}{8\sqrt{2}}(\sqrt{5}+1) - \frac{\sqrt{3}-1}{8}\sqrt{5-\sqrt{5}}.$$

$$\sin 42^\circ = -\tfrac{1}{8}(\sqrt{5}-1) + \frac{\sqrt{3}}{4\sqrt{2}}\sqrt{5+\sqrt{5}}.$$

$$\sin 45^\circ = \frac{1}{\sqrt{2}}.$$

$$\sin 48^\circ = \frac{\sqrt{3}}{8}(\sqrt{5}-1) + \frac{1}{4\sqrt{2}}\sqrt{5+\sqrt{5}}.$$

$$\sin 51^\circ = \frac{\sqrt{3}-1}{8\sqrt{2}}(\sqrt{5}+1) + \frac{\sqrt{3}+1}{8}\sqrt{5-\sqrt{5}}.$$

$$\sin 54^\circ = \tfrac{1}{4}(\sqrt{5}+1).$$

$$\sin 57^\circ = -\frac{\sqrt{3}-1}{8\sqrt{2}}(\sqrt{5}-1) + \frac{\sqrt{3}+1}{8}\sqrt{5+\sqrt{5}}.$$

$$\sin 60^\circ = \tfrac{1}{2}\sqrt{3}.$$

$$\sin 63^\circ = \frac{1}{4\sqrt{2}}(\sqrt{5}-1) + \tfrac{1}{4}\sqrt{5+\sqrt{5}}.$$

$$\sin 66^\circ = \tfrac{1}{8}(\sqrt{5}+1) + \frac{\sqrt{3}}{4\sqrt{2}}\sqrt{5-\sqrt{5}}.$$

$$\sin 69^\circ = \frac{\sqrt{3}+1}{8\sqrt{2}}(\sqrt{5}+1) + \frac{\sqrt{3}-1}{8}\sqrt{5-\sqrt{5}}.$$

$$\sin 72^\circ = \frac{1}{2\sqrt{2}}\sqrt{5+\sqrt{5}}.$$

$$\sin 75^\circ = \frac{1}{2\sqrt{2}}(\sqrt{3}+1).$$

$$\sin 78^\circ = \tfrac{1}{8}(\sqrt{5}-1) + \frac{\sqrt{3}}{4\sqrt{2}}\sqrt{5+\sqrt{5}}.$$

$$\sin 81^\circ = \frac{1}{4\sqrt{2}}(\sqrt{5}+1) + \tfrac{1}{4}\sqrt{5-\sqrt{5}}.$$

$$\sin 84^\circ = \frac{\sqrt{3}}{8}(\sqrt{5}+1) + \frac{1}{4\sqrt{2}}\sqrt{5-\sqrt{5}}.$$

$$\sin 87^\circ = \frac{\sqrt{3}-1}{8\sqrt{2}}(\sqrt{5}-1) + \frac{\sqrt{3}+1}{8}\sqrt{5+\sqrt{5}}.$$

$$\sin 90^\circ = 1. \qquad (C.\ 304.)$$

(44.) *Lengths of arcs in terms of radius, and their logarithms.*

$$
\begin{aligned}
1^\circ &= 0{,}017453292520; & \log_{10} 1^\circ + 10 &= 8{,}2418773675.\\
1' &= 0{,}000290888209; & \log_{10} 1' + 10 &= 6{,}4637261171.\\
1'' &= 0{,}000004848137; & \log_{10} 1'' + 10 &= 4{,}6855748667.\\
1^g &= 0{,}0157079632679; & \log_{10} 1^g + 10 &= 8{,}1961198769.
\end{aligned}
$$

$\log_{10} \pi = 0{,}4971498726.$ (*Enc. Met.* V. I. p. 672.)

(45.) *Series for the construction of tables,*

$$
\begin{aligned}
\sin \frac{m}{n} \cdot \frac{\pi}{2} = {} & \frac{m}{n} \times 1{,}570796326794827 && - \frac{m^3}{n^3} \times 0{,}645964097506246\\
& + \frac{m^5}{n^5} \times 0{,}079692626246167 && - \frac{m^7}{n^7} \times 0{,}004681754135319\\
& + \frac{m^9}{n^9} \times 0{,}000160441184787 && - \frac{m^{11}}{n^{11}} \times 0{,}000003598843235\\
& + \frac{m^{13}}{n^{13}} \times 0{,}000000056921729 && - \frac{m^{15}}{n^{15}} \times 0{,}000000000668804\\
& + \frac{m^{17}}{n^{17}} \times 0{,}000000000006067 && - \frac{m^{19}}{n^{19}} \times 0{,}000000000000044.
\end{aligned}
$$

$$
\begin{aligned}
\cos \frac{m}{n} \cdot \frac{\pi}{2} = 1 & - \frac{m^2}{n^2} \times 1{,}233700550136170 && + \frac{m^4}{n^4} \times 0{,}253669507901048\\
& - \frac{m^6}{n^6} \times 0{,}020863480763353 && + \frac{m^8}{n^8} \times 0{,}000919260274839\\
& - \frac{m^{10}}{n^{10}} \times 0{,}000025202042373 && + \frac{m^{12}}{n^{12}} \times 0{,}000000471087478\\
& - \frac{m^{14}}{n^{14}} \times 0{,}000000006386603 && + \frac{m^{16}}{n^{16}} \times 0{,}000000000065660\\
& - \frac{m^{18}}{n^{18}} \times 0{,}000000000000529 && + \frac{m^{20}}{n^{20}} \times 0{,}000000000000003.
\end{aligned}
$$

A few terms of these series will generally be sufficient, as it will never be necessary to take $\frac{m}{n} > \frac{1}{2}$.

The sines of the lesser divisions, as minutes, are usually found by the method of differences, which will be explained hereafter.

Series for calculating the tangent and cotangent, independently of the sine and cosine.

$$\tan\frac{m}{n}\cdot\frac{\pi}{2}=\frac{2mn}{n^2-m^2}\times 0{,}6366197723675813$$

$$
\begin{aligned}
&+\frac{m}{n}\times 0{,}297556782059734 && +\frac{m^3}{n^3}\times 0{,}018688650277330\\
&+\frac{m^5}{n^5}\times 0{,}001842475203510 && +\frac{m^7}{n^7}\times 0{,}000197580071520\\
&+\frac{m^9}{n^9}\times 0{,}000021697737325 && +\frac{m^{11}}{n^{11}}\times 0{,}000002401136991\\
&+\frac{m^{13}}{n^{13}}\times 0{,}000000266413303 && +\frac{m^{15}}{n^{15}}\times 0{,}000000029586468\\
&+\frac{m^{17}}{n^{17}}\times 0{,}000000003286788 && +\frac{m^{19}}{n^{19}}\times 0{,}000000000365175\\
&+\frac{m^{21}}{n^{21}}\times 0{,}000000000040754 && +\frac{m^{23}}{n^{23}}\times 0{,}000000000004508\\
&+\frac{m^{25}}{n^{25}}\times 0{,}000000000000501 && +\frac{m^{27}}{n^{27}}\times 0{,}000000000000056.
\end{aligned}
$$

$$\cot\frac{m}{n}\cdot\frac{\pi}{2}=\frac{n}{m}\times 0{,}6366197723675813$$

$$-\frac{4mn}{4n^2-m^2}\times 0{,}3183098861837907$$

$$
\begin{aligned}
&-\frac{m}{n}\times 0{,}205288889414508 && -\frac{m^3}{n^3}\times 0{,}006551074788218\\
&-\frac{m^5}{n^5}\times 0{,}00084529255397 && -\frac{m^7}{n^7}\times 0{,}000020279106052\\
&-\frac{m^9}{n^9}\times 0{,}000001236652718 && -\frac{m^{11}}{n^{11}}\times 0{,}000000076495882\\
&-\frac{m^{13}}{n^{13}}\times 0{,}000000004759738 && -\frac{m^{15}}{n^{15}}\times 0{,}000000000296905\\
&-\frac{m^{17}}{n^{17}}\times 0{,}000000000018541 && -\frac{m^{19}}{n^{19}}\times 0{,}000000000001158\\
&-\frac{m^{21}}{n^{21}}\times 0{,}000000000000072 && -\frac{m^{23}}{n^{23}}\times 0{,}000000000000005.
\end{aligned}
$$

(46.) *Construction of logarithmic tables.*

$$\log_{10}\sin\frac{m}{n}\cdot\frac{\pi}{2}=\log_{10}m+\log_{10}(2n-m)+\log_{10}(2n+m)-3\log_{10}m$$

$$+9{,}594059885702190$$

$$
\begin{aligned}
&-\frac{m^2}{n^2}\times 0{,}070022826605902 && -\frac{m^4}{n^4}\times 0{,}001117266441662\\
&-\frac{m^6}{n^6}\times 0{,}000039229146454 && -\frac{m^8}{n^8}\times 0{,}000001729270798\\
&-\frac{m^{10}}{n^{10}}\times 0{,}000000084362986 && -\frac{m^{12}}{n^{12}}\times 0{,}000000004348716
\end{aligned}
$$

$$-\frac{m^{14}}{n^{14}}\times 0{,}000000000231931 \qquad -\frac{m^{16}}{n^{16}}\times 0{,}000000000012659$$

$$-\frac{m^{18}}{n^{18}}\times 0{,}000000000000703 \qquad -\frac{m^{20}}{n^{20}}\times 0{,}000000000000040.$$

$$\log_{10}\cos\frac{m}{n}\cdot\frac{\pi}{2} = \log_{10}(n-m) + \log_{10}(n+m) - 2\log_{10} n$$

$$-\frac{m^{2}}{n^{2}}\times 0{,}101494859341893 \qquad -\frac{m^{4}}{n^{4}}\times 0{,}003187294065451$$

$$-\frac{m^{6}}{n^{6}}\times 0{,}000209485800017 \qquad -\frac{m^{8}}{n^{8}}\times 0{,}000016848348598$$

$$-\frac{m^{10}}{n^{10}}\times 0{,}000001480193987 \qquad -\frac{m^{12}}{n^{12}}\times 0{,}000000186502272$$

$$-\frac{m^{14}}{n^{14}}\times 0{,}000000012981715 \qquad -\frac{m^{16}}{n^{16}}\times 0{,}000000001261471$$

$$-\frac{m^{18}}{n^{18}}\times 0{,}000000000124567 \qquad -\frac{m^{20}}{n^{20}}\times 0{,}000000000012456$$

$$-\frac{m^{22}}{n^{22}}\times 0{,}000000000001258 \qquad -\frac{m^{24}}{n^{24}}\times 0{,}000000000000128.$$

If m is small, $\log\left(1-\frac{m^2}{n^2}\right)$ may be expanded into the series

$$-2m\left\{\frac{m^2}{2n^2-m^2}+\frac{1}{3}\left(\frac{m^2}{2n^2-m^2}\right)^3+\&c.\right\}$$

which will render the calculation independent of logarithmic tables.

If we find the logarithmic cosines of arcs < 45°, and the logarithmic sines of arcs between 45°, and 90°, the rest may be found from the formula

$$\log_{10}\sin a = \log_{10}\sin 2a - \log_{10}\cos a + 9{,}6989700043360 19.$$

(*Enc. Met. Trig.* §. 10.)

(47.) *Logarithmic series for the sine, cosine, and tangent.*

$$\log_e \sin x = \log_e x - \left\{\frac{x^2}{2.3}+\frac{x^4}{2^2.3^2.5}+\frac{x^6}{3^4.5.7}+\frac{x^8}{2^3.3^3.5^2.7}\right.$$

$$+\frac{x^{10}}{3^4.5^2.7.11}+\frac{691\,x^{12}}{2.3^7.5^3.7^2.11.13}+\frac{2x^{14}}{3^6.5^2.7^2.11.13}$$

$$\left.+\frac{3617\,x^{16}}{2^4.3^7.5^4.7^2.11.13.17}+\&c.\right\}$$

$$\log_e \cos x = -\left\{\frac{x^2}{2} + \frac{x^4}{2^2.3} + \frac{x^6}{3^2.5} + \frac{17x^8}{2^3.3^2.5.7} + \frac{31x^{10}}{3^4.5^2.7}\right.$$

$$\left. + \frac{691x^{12}}{2.3^5.5^2.7.11} + \frac{10922x^{14}}{3^5.5^2.7^2.11.13} + \frac{929569x^{16}}{2^4.3^6.5^3.7^2.11.13} + \&c.\right\}$$

$$\log_e \tan x = \log_e x + \left\{\frac{x^2}{3} + \frac{7x^4}{2.3^2.5} + \frac{62x^6}{3^4.5.7} + \frac{127x^8}{2^2.3^3.5^2.7}\right.$$

$$+ \frac{146x^{10}}{3^5.5^2.11} + \frac{1414477x^{12}}{3^7.5^3.7^2.11.13} + \frac{32764x^{14}}{3^6.5^2.7^2.11.13}$$

$$\left. + \frac{16931177x^{16}}{2.3^7.5^4.7.11.13.17} + \&c.\right\} \qquad (C.\ 405-7.)$$

(48.) To find the logarithmic sines and tangents of small arcs.

$$\log_{10} \text{Sin}\, x = \log_{10} n + 4.6855749 - \tfrac{1}{3}(10 - \log_{10} \text{Cos}\, x);$$

$$\log_{10} \text{Tan}\, x = \log_{10} n + 4.6855749 + \tfrac{2}{3}(10 - \log_{10} \text{Cos}\, x);$$

n being the value of x in seconds and decimal parts of seconds.

(*Taylor, Log. Introd.* p. 17.)

(49.) *Formulæ for the verification of tables.*

$$\sin x + \sin(36° - x) + \sin(72° + x) = \sin(36° + x) + \sin(72° - x).$$

(*Euler, Anal. Inf.* V. I. p. 201.)

$$\sin(90° - x) = \sin(54° + x) + \sin(54° - x)$$
$$- \sin(18° + x) - \sin(18° - x). \qquad (Leg.\ 40.)$$

$$\cos(36° + x) + \cos(36° - x) = \cos x + \cos(72° + x) + \cos(72° - x):$$

this is only a particular case of the more general theorem

$$\left.\begin{array}{l}\cos x + \cos\left(2\frac{\pi}{n} + x\right) + \cos\left(4\frac{\pi}{n} + x\right) + \&c. \\ \quad + \cos\left(2\frac{\pi}{n} - x\right) + \cos\left(4\frac{\pi}{n} - x\right) + \&c.\end{array}\right\}$$

$$= \begin{cases} \cos\left(\frac{\pi}{n} + x\right) + \cos\left(3\frac{\pi}{n} + x\right) + \&c. \\ + \cos\left(\frac{\pi}{n} - x\right) + \cos\left(3\frac{\pi}{n} - x\right) + \&c. \end{cases}$$

in which each series is to be continued until the angle attains its greatest value next below 90°. (*Enc. Met.* V. I. p. 695.)

TRIGONOMETRICAL SOLUTION OF EQUATIONS.

(50.) *Quadratic equations.*

[1] $x^2 - px + q = 0.$

Assume $(\sin\theta)^2 = \frac{4q}{p^2}$; then

$$x = p\,(\sin\tfrac{1}{2}\theta)^2, \text{ or } p\,(\cos\tfrac{1}{2}\theta)^2;$$

$$\text{or } x = \sqrt{q}\,.\tan\tfrac{1}{2}\theta, \text{ or } \sqrt{q}\,.\cot\tfrac{1}{2}\theta.$$

[2] $x^2 + px - q = 0.$

Assume $(\tan\theta)^2 = \frac{4q}{p^2}$; then

$$x = \tfrac{1}{2}p\tan\theta.\tan\tfrac{1}{2}\theta, \text{ or } -\tfrac{1}{2}p\tan\theta.\cot\tfrac{1}{2}\theta;$$

$$\text{or } x = \sqrt{q}\,\tan\tfrac{1}{2}\theta, \text{ or } -\sqrt{q}\,\cot\tfrac{1}{2}\theta.$$

The equations $x^2 + px + q$, and $x^2 - px - q = 0$, have respectively the same roots as the above forms, but with contrary signs. (*C.* 810—23; *L.* 426; *Hind, Trig.* 285.)

(51.) *Cubic equations.*

[1] $x^3 + qx \mp r = 0.$

Assume $\tan\theta = \frac{2}{r}\left(\frac{q}{3}\right)^{\frac{3}{2}}$, and $\tan\phi = (\tan\tfrac{1}{2}\theta)^{\frac{1}{3}}$, then

$$x = \pm 2\,(\tfrac{1}{3}q)^{\frac{1}{2}}.\cot 2\phi.$$

[2] $x^3 - qx \mp r = 0$, and $4q^3 < 27r^2$.

Assume $\sin\theta = \frac{2}{r}\left(\frac{q}{3}\right)^{\frac{3}{2}}$, and $\tan\phi = (\tan\tfrac{1}{2}\theta)^{\frac{1}{3}}$, then

$$x = \pm\frac{2\,(\tfrac{1}{3}q)^{\frac{1}{2}}}{\sin 2\phi}.$$

[3] $x^3 - qx \pm r = 0$, and $4q^3 \not< 27r^2$.

Assume $\sin 3\theta = \frac{r}{2}\left(\frac{3}{q}\right)^{\frac{3}{2}}$, the three values of x are

$$\begin{aligned} x &= \pm 2\left(\tfrac{1}{3}q\right)^{\frac{1}{2}} . \sin\theta, \\ &= \pm 2\left(\tfrac{1}{3}q\right)^{\frac{1}{2}} . \sin(60^\circ - \theta), \\ &= \mp 2\left(\tfrac{1}{3}q\right)^{\frac{1}{2}} . \sin(60^\circ + \theta). \end{aligned}$$

(*C.* 824—45; *L.* 427; *Hind*, 286—90.)

(52.) *Solution of the equation* $x^n \pm 1 = 0$.

$$\begin{aligned} x^n - 1 = (x-1) &\left\{\left(x - \cos\frac{2}{n}\pi + \sqrt{-1}.\sin\frac{2}{n}\pi\right)\right. \\ &\quad \left.\left(x - \cos\frac{2}{n}\pi - \sqrt{-1}.\sin\frac{2}{n}\pi\right)\right\}\ldots \\ &\left\{\left(x - \cos\frac{n-1}{n}\pi + \sqrt{-1}.\sin\frac{n-1}{n}\pi\right)\right. \\ &\quad \left.\left(x - \cos\frac{n-1}{n}\pi - \sqrt{-1}.\sin\frac{n-1}{n}\pi\right)\right\}, \end{aligned}$$

$$\begin{aligned} &= (x-1)\left(x^2 - 2\cos\frac{2}{n}\pi.x + 1\right)\left(x^2 - 2\cos\frac{4}{n}\pi.x + 1\right)\ldots \\ &\qquad \left(x^2 - 2\cos\frac{n-1}{n}\pi.x + 1\right); \text{ if } n \text{ is odd:} \\ &= (x^2-1)\left(x^2 - 2\cos\frac{2}{n}\pi.x + 1\right)\left(x^2 - 2\cos\frac{4}{n}\pi.x + 1\right)\ldots \\ &\qquad \left(x^2 - 2\cos\frac{n-2}{n}\pi.x + 1\right); \text{ if } n \text{ is even.} \end{aligned}$$

$$\begin{aligned} x^n + 1 &= (x+1)\left(x^2 - 2\cos\frac{1}{n}\pi.x + 1\right)\left(x^2 - 2\cos\frac{3}{n}\pi.x + 1\right)\ldots \\ &\qquad \left(x^2 - 2\cos\frac{n-2}{n}\pi.x + 1\right); \text{ if } n \text{ is odd:} \\ &= \left(x^2 - 2\cos\frac{1}{n}\pi.x + 1\right)\left(x^2 - 2\cos\frac{3}{n}\pi.x + 1\right)\ldots \\ &\qquad \left(x^2 - 2\cos\frac{n-1}{n}\pi.x + 1\right); \text{ if } n \text{ is even.} \end{aligned}$$

(*G. A.* ch. xi.)

(53.) *Solution of the equation* $x^{2n} - 2\cos\theta . x^n + 1 = 0.$

$$x^{2n} - 2\cos\theta . x^n + 1 = (x^2 - 2\cos\frac{\theta}{n}\, x + 1)$$
$$(x^2 - 2\cos\frac{2\pi+\theta}{n}\, x + 1) \dots (x^2 - 2\cos\frac{2(n-1)\pi+\theta}{n}\, x + 1).$$

$$x^{2n} + 2\cos\theta . x^n + 1 = (x^2 - 2\cos\frac{\pi+\theta}{n}\, x + 1)$$
$$(x^2 - 2\cos\frac{3\pi+\theta}{n}\, x + 1) \dots (x^2 - 2\cos\frac{(2n-1)\pi+\theta}{n}\, x + 1).$$

(*Hind*, 291, 2; *G. A.* 71.)

Properties of a Quadrilateral inscribed in a Circle.

(54.) Let A, B, C, D, be the angles;

a, b, c, d, the sides AB, BC, CD, DA, respectively;

α, β, the diagonals AC, BD;

ϕ their angle of intersection;

$s = \frac{1}{2}(a+b+c+d)$: then

$$\cos A = \frac{a^2+d^2-b^2-c^2}{2(ad+bc)}.$$

$$\sin A = \frac{2}{ad+bc} . \left\{(s-a)(s-b)(s-c)(s-d)\right\}^{\frac{1}{2}};$$

$$\sin\tfrac{1}{2}A = \left\{\frac{(s-a)(s-d)}{ad+bc}\right\}^{\frac{1}{2}}.$$

$$\cos\tfrac{1}{2}A = \left\{\frac{(s-b)(s-c)}{ad+bc}\right\}^{\frac{1}{2}}.$$

$$\tan\tfrac{1}{2}A = \left\{\frac{(s-a)(s-d)}{(s-b)(s-c)}\right\}^{\frac{1}{2}}.$$

$$\alpha = \left\{(ac+bd)\frac{ad+bc}{ab+cd}\right\}^{\frac{1}{2}} : \qquad \beta = \left\{(ac+bd)\frac{ab+cd}{ad+bc}\right\}^{\frac{1}{2}}.$$

The *area* $= \left\{(s-a)(s-b)(s-c)(s-d)\right\}^{\frac{1}{2}}$;

$= \frac{1}{2}\alpha\beta . \sin\phi.$

$$\sin\phi = \frac{2}{ac+bd} . \left\{(s-a)(s-b)(s-c)(s-d)\right\}^{\frac{1}{2}}.$$

The radius of the circumscribed circle

$$=\tfrac{1}{4}\cdot\left.\frac{\overline{(ab+cd)(ac+bd)(ad+bc)}}{(s-a)(s-b)(s-c)(c-d)}\right|^{\frac{1}{2}}.$$

(*Hind*, *Trig*. 166—74; *Leg*. *Geom*. *Note* v.)

PROPERTIES OF POLYGONS.

(55.) Let A_1, A_2, &c. A_n, be the exterior angles, a_1, a_2, &c. a_n, the sides A_1A_2, A_2A_3, &c. A_nA_1, respectively:

$$\begin{aligned}2\,area = a_1\{&a_2.\sin A_2 + a_3.\sin(A_2+A_3) + \&c.\\ &+ a_n.\sin(A_2+A_3+\&c.+A_n)\}\\ &+ a_2\{a_3.\sin A_3 + \&c. + a_n.\sin(A_3+\&c.+A_n)\}\\ &+ \&c. + a_{n-1}a_n.\sin A_n.\end{aligned}$$ [a] (*Lhuilier*, *Polygonom*. VIII.)

$$a_1 = a_2.\cos a_1,a_2 + a_3.\cos a_1,a_3 + \&c. + a_n.\cos a_1,a_n.$$

$$a_1^2 = a_2^2+a_3^2+\&c.+a_n^2-2\{a_2a_3.\cos a_2,a_3 + a_2a_4.\cos a_2,a_4 + \&c.\}.$$

(*Hamilton*, *Analytical Geometry*, 41.)

(56.) Let $a_1 = a_2 = \&c. = a_n$,

R_1, the radius of the circumscribed circle,

R_2, that of the circumscribed circle: then

$$A = \frac{n-2}{n}\pi; \qquad a = 2R_1\tan\frac{\pi}{n} = 2R_2\sin\frac{\pi}{n}.$$

$$R_1 = \tfrac{1}{2}a\tan\frac{n-2}{2n}\pi; \quad R_2 = \tfrac{1}{2}a\sec\frac{n-2}{2n}\pi.$$

$$\begin{aligned}\text{The } area &= \frac{na^2}{4}\tan\frac{n-2}{2n}\pi\\ &= \tfrac{1}{2}naR_1 = \tfrac{1}{4}na(4R_2^2-a^2)^{\frac{1}{2}}\\ &= nR_1^2.\tan\frac{\pi}{n} = \tfrac{1}{2}nR_2^2.\sin\frac{2\pi}{n}.\end{aligned}$$

(*Hind*, *Trig*. 175—82.)

---

[a] $2\,area = \overset{n-1}{\mathrm{S}}_m a_m.\overset{n-m}{\mathrm{S}}_r a_{m+r}.\sin\overset{r}{\mathrm{S}}_s A_{m+s}.$

ANALYTICAL GEOMETRY.

(1.) METHOD of representing algebraical quantities geometrically. (*H. A. G. Introd.* 6—10; *G. G. A.* 1—6; *Biot*, Ch. i.)

(2.) Analytical solution of determinate geometrical problems. (*H. A. G.* 14—30; *G. G. A.* 7; *Biot*, Ch. ii.)

(3.) Relation of indeterminate equations to Geometry; and definitions. (*H. A. G.* 43—8; *Biot*, 29—40; *L. A. G.* §. 2.)

ANALYTICAL GEOMETRY OF TWO DIMENSIONS.

(4.) *The straight line.* The equation to a straight line is

$$y = ax + b.$$

Let the line be represented by l, then if the co-ordinates are oblique, $a = \dfrac{\sin l,x}{\sin l,y}$;

if rectangular, $a = \tan l,x$.

The equation to a straight line may be put under the form

$$\frac{x}{a} + \frac{y}{b} = 1.$$

The equation in terms of p, the perpendicular from the origin is $x.\cos p,x + y.\cos p,y = p$.

(*H. A. G.* 49—51; *G. G. A.* Ch. ii; *Biot*, 41—7; *L. A. G.* §. 3.)

(5.) The equation to a straight line passing through the point (x_1,y_1) is $y - y_1 = a(x - x_1)$.

If the line passes through two points, (x_1,y_1), (x_2,y_2), the equation is $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$.

The co-ordinates of the intersection of two lines

$$y = a_1 x + b_1, \qquad y = a_2 x + b_2;$$

are $$x = \frac{b_2 - b_1}{a_1 - a_2}, \qquad y = \frac{a_1 b_2 - a_2 b_1}{a_1 - a_2}.$$

If a third line, $y = a_3 x + b_3$, passes through the point of intersection, then

$$(a_1 b_2 - a_2 b_1) - (a_1 b_3 - a_3 b_1) + (a_2 b_3 - a_3 b_2) = 0.$$

Let the two given lines be represented by l_1, l_2; then

$$\tan l_1, l_2 = \sin x, y \, \frac{a_1 - a_2}{1 + (a_1 + a_2) \cos x, y + a_1 a_2}.$$

If the two lines are perpendicular to each other,

$$1 + (a_1 + a_2) \cos x, y + a_1 a_2 = 0.$$

If the co-ordinates are rectangular, then

$$\tan l_1, l_2 = \frac{a_1 - a_2}{1 + a_1 a_2},$$

$$\sin l_1, l_2 = \frac{a_1 - a_2}{\overline{(1 + a_1^2)(1 + a_2^2)}\,|^{\frac{1}{2}}},$$

$$\cos l_1, l_2 = \frac{1 + a_1 a_2}{\overline{(1 + a_1^2)(1 + a_2^2)}\,|^{\frac{1}{2}}}.$$

If in this case the lines are perpendicular to each other,

$$1 + a_1 a_2 = 0.$$

The distance between two points, (x_1, y_1), (x_2, y_2),

$$= \overline{(x_1 - x_2)^2 + (y_1 - y_2)^2}\,|^{\frac{1}{2}}.$$

Let p be the perpendicular from a given point, (x_1, y_1), on the line $y = ax + b$, then

$$p = \pm \frac{y_1 - a x_1 - b}{(1 + a^2)^{\frac{1}{2}}}.$$

If (x_1, y_1), (x_2, y_2), (x_3, y_3), are the angular points of a triangle,

$$Area = \tfrac{1}{2} \{ (x_1 y_2 - x_2 y_1) - (x_1 y_3 - x_3 y_1) + (x_2 y_3 - x_3 y_2) \}.$$

(*H. A. G.* 52—9; *H. C. S.* Ch. ii.; *Biot,* 48—53; *L. A. G.* §. 4.)

(6.) Let x, y, be the co-ordinates of any line,

x_1, y_1, the co-ordinates of the same line in another system having the same origin, then

$$x = \frac{1}{\sin x,y}(x_1.\sin x_1,y + y_1.\sin y_1,y),$$
$$y = \frac{1}{\sin x,y}(x_1.\sin x_1,x + y_1.\sin y_1,x).$$

If the axes in the original system are rectangular, and in the new system, oblique, the above formulæ become

$$x = x_1 \cos x_1,x + y_1 \cos y_1,x,$$
$$y = x_1 \sin x_1,x + y_1 \sin y_1,x.$$

If the original axes are oblique, and the new ones rectangular, then

$$x = \frac{1}{\sin x,y}(x_1 \sin x_1,y + y_1 \cos x_1,y),$$
$$y = \frac{1}{\sin x,y}(x_1 \sin x_1,x + y_1 \cos x_1,x).$$

If both systems are rectangular, then

$$x = x_1 \cos x_1,x - y_1 \sin x_1,x,$$
$$y = x_1 \sin x_1,x + y_1 \cos x_1,x.$$

If it be required to change the origin, as well as the direction of the co-ordinates, then a, b, the co-ordinates of the new origin, must be respectively added to the values of x and y in the above formulæ.

If the origin alone be changed, the direction of the co-ordinates remaining the same, then

$$x = a + x_1,$$
$$y = b + y_1.$$

(*H. A. G.* 71–6; *L. A. G.* §. vi; *G. G. A.* Ch. iii; *Biot*, 86–96.)

(7.) *Transformation of polar co-ordinates.*

Let a, b, be the co-ordinates of the new pole, u, the radius vector, and α the angle which the axis to which u is referred makes with the axis of x, then

$$x = a + u\frac{\sin\{x,y - (\theta + \alpha)\}}{\sin x,y},$$

$$y = b + u\frac{\sin(\theta + \alpha)}{\sin x,y}.$$

If the axes are rectangular, these formulæ become

$$x = a + u\cos(\theta + \alpha),$$
$$y = b + u\sin(\theta + \alpha).$$

(*H. A. G.* 77—80; *Biot*, 100.)

The Circle.

(8.) Let α, β, be the co-ordinates of the centre, r the radius; then the equation to the circle is

$$(x - \alpha)^2 + (y - \beta)^2 + 2(x - \alpha)(y - \beta).\cos x,y = r^2.$$

The equation between rectangular co-ordinates is

$$(x - \alpha)^2 + (y - \beta)^2 = r^2. \qquad (1)$$

If the axis of x, or y, passes through the centre, the equation becomes respectively,

$$\left.\begin{aligned} x^2 + (y - \beta)^2 &= r^2, \\ \text{or } (x - \alpha)^2 + y^2 &= r^2. \end{aligned}\right\} \qquad (2)$$

If the origin is in the circumference, (1), (2), become respectively,

$$x^2 + y^2 - 2\alpha x - 2\beta y = 0; \qquad (3)$$

$$\left.\begin{aligned} y^2 &= 2rx - x^2, \\ x^2 &= 2ry - y^2. \end{aligned}\right\} \qquad (4)$$

If the origin is at the centre, then

$$x^2 + y^2 = r^2. \qquad (5)$$

of these forms, (4) and (5) occur most frequently.

The general form when referred to rectangular co-ordinates is
$$y^2+x^2+Ay+Bx+C=0.$$

(9.) *The polar equation.* Let a, b, be the co-ordinates of the pole, u the radius vector, and the angle $u,x=\theta$;
the polar equation is
$$u^2\pm 2(a\cos\theta+b\sin\theta)u+a^2+b^2-r^2=0;$$
$+$ or $-$, acting as the pole is situated within or without the circle. (*H. A. G.* Ch. v; *G. G. A.* Ch. x; *Biot* 105—14.)

(10.) The intersection of the circle $x^2+y^2=r^2$ and the straight line $y=ax+b$ may be determined from the equation
$$y_1^2-\frac{2b}{a^2+1}y_1+\frac{b^2-a^2r^2}{a^2+1}=0,$$
which having only two roots, it follows that a straight line cannot cut a circle in more than two points.

The equation to a tangent at a given point (x_1,y_1) in the circumference, is
$$y-y_1=-\frac{x_1+y_1\cos x,y}{y_1+x_1\cos x,y}(x-x_1),$$
if the axes are rectangular,
$$y-y_1=-\frac{x_1}{y_1}(x-x_1), \text{ or}$$
$$xx_1+yy_1=r^2.$$

To draw a tangent from a given point (x_2,y_2) without the circle: the co-ordinates x_1, y_1, of the point of contact may be determined by the solution of the equations,
$$x_2x_1+y_2y_1=r^2,$$
$$x_1^2+y_1^2=r^2.$$

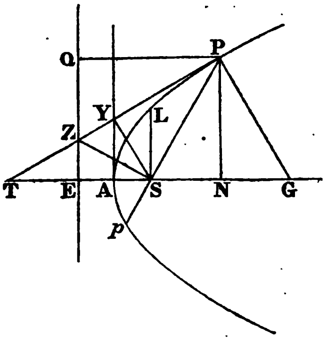
(*H. A. G.* Ch. vi; *H. C. S.* 37—43; *Biot*, 115—23.)

CONIC SECTIONS.

THE PARABOLA.

(11.) *The parabola referred to its axis.*

Let S be the *focus*, P any point in the curve, and PQ perpendicular to EQ the *directrix*, then $PS=PQ$.



Let PT be a tangent at the point P, and PG the *normal*.

If the origin is at the vertex, and the axis coincides with the axis of x, the equation is

$$y^2=4mx,$$

$$\text{or } PN^2=4AS.AN.$$

$\angle QPT=\angle SPT.$

$SP=AN+AS=ST=SG.$

The intersection of the straight line $y=ax+b$,

and the parabola $y^2=4mx$;

may be determined by the solution of the equation

$$y^2-\frac{4m}{a}y+\frac{4mb}{a}=0:$$

which has only two roots; hence a straight line cannot cut a parabola in more than two points.

The equation to a tangent at the point (x_1,y_1) is

$$y-y_1=\frac{2m}{y_1}(x-x_1),$$

$$\text{or } yy_1=2m(x+x_1).$$

The *subtangent*, $NT = 2AN$.

The equation to the normal is

$$y - y_1 = -\frac{y_1}{2m}(x - x_1).$$

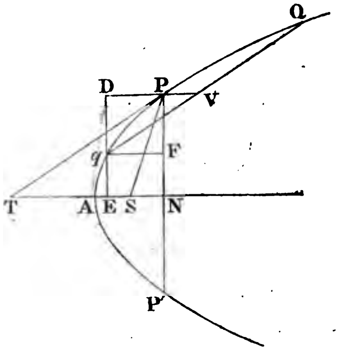
The subnormal, $NG = 2AS = 2m$.

To draw a tangent to the parabola from a given point (a,b) without it: x_1, y_1 the co-ordinates of the point of contact may be determined from the equations

$$b = \frac{2m}{y_1}(a + x_1),$$

$$y_1^2 = 4mx_1. \qquad \text{(}H. C. S.\ 46\text{—}62;\ Hust.\ \text{Prop. 3—7.)}$$

(12.) *The parabola referred to the focus.*



The polar equation, the focus being the pole, is

$$u = \frac{2m}{1 + \cos\theta} = \frac{m}{(\sin\frac{1}{2}\theta)^2}.$$

$$\frac{1}{SP} + \frac{1}{Sp} = \frac{2}{SL}. \qquad \text{(}See\ Fig.\ \text{p. 160.)}$$

$$SP . Sp = AS . Pp.$$

The tangent at any point, and the perpendicular on it from the focus intersect the axis AY in the same point.

$$SP . SA = SY^2.$$

Let Z be the intersection of the tangent with the directrix, SZ is perpendicular to SP. (*H. C. S.* 63—71; *Hust.* Prop. 8.)

(13.) *The parabola referred to any diameter.* A diameter and the tangent at its vertex being the axes, the equation is

$$y^2 = 4m_1 . x,$$

$$\text{or } QV^2 = 4SP.PV.$$

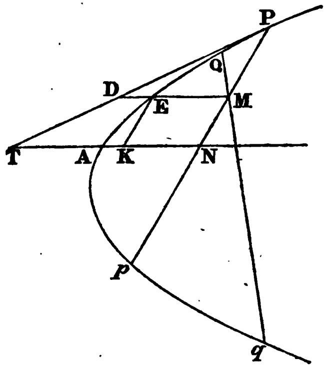
The parameter $= 4SP$.

$2m.qF = PF.FP'$.

$PF^2 = 2m.PV$.

If $4m_1$, $4m_2$ are the parameters corresponding to the ordinates Pp, Qq respectively, then

$$4m_1 . EM = PM.Mp.$$



$PM.Mp : QM.Mq :: m_1 : m_2$.

$PM : Mp :: DE : EM$.

The subtangent NT is bisected by the curve at A.

If from the several points of any line given in position pairs of tangents be drawn to a parabola, the chords-joining the corresponding points of contact will pass through the same point.

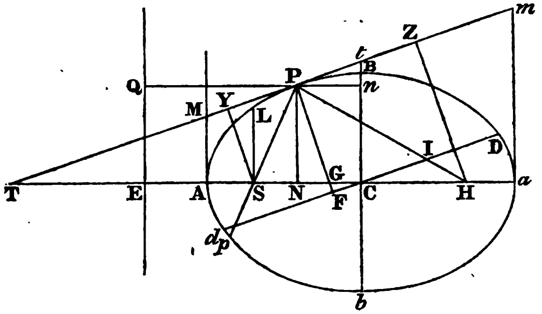
The *chord of curvature* through the focus $= 4SP$.

The *radius of curvature* $= \dfrac{2SP^{\frac{3}{2}}}{SA^{\frac{1}{2}}}$.

(*H. C. S.* 71—84; *Hust.* Prop. 9—15; *Biot*, 191—223; *G. G. A.* Ch. xii.)

THE ELLIPSE.

(14.) *The Ellipse referred to its axes.* Let S be the focus, and



PQ perpendicular to EQ the directrix; then

$$SP : PQ :: e : 1;$$

e being a constant quantity, and < 1,

The equation referred to the axis, and a tangent at the vertex is

$$y^2 = \frac{b^2}{a^2}(2ax - x^2),$$

$$AN.Na : NP^2 :: AC^2 : BC^2.$$

The equation referred to the centre and axes is

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2); \text{ or}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

$$AS.Sa = BC^2 = AC.SL.$$

The intersection of an ellipse with the straight line,

$$y = \alpha x + \beta,$$

may be determined from the equation

$$y^2 - \frac{2b^2\beta}{a^2\alpha^2 + b^2}y + \frac{b^2(\beta^2 - a^2\alpha^2)}{a^2\alpha^2 + b^2} = 0;$$

which has only two roots, hence a straight line cannot cut an ellipse in more than two points.

The equation to a tangent at the point (x_1,y_1) is

$$y-y_1=-\frac{b^2}{a^2}\cdot\frac{y_1}{x_1}(x-x_1),\text{ or}$$

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.$$

Let PT be a tangent at the point P, then

$$CN\,.\,CT=AC^2;\qquad Cn\,.\,Ct+BC^2.$$

$PF\,.\,PG=BC^2.$

The subtangent, $NT=\dfrac{a^2-x_1^{\,2}}{x_1}.$

If the ordinate NP be produced to meet the circumference of a circle described on Aa in R, the tangents at the points P and R will meet the axis produced in the same point.

The equation to the normal PG is

$$y-y_1=\frac{a^2}{b^2}\cdot\frac{y_1}{x_1}(x-x_1).$$

The subnormal $NG=\dfrac{BC^2}{AC^2}\cdot CN.$

To draw a tangent to an ellipse from a given point (x_2,y_2) without it: the point of contact (x_1,y_1) must be determined from the equations

$$\frac{y_2y_1}{b^2}+\frac{x_2x_1}{a^2}=1,$$

$$\frac{y_1^{\,2}}{b^2}+\frac{x_1^{\,2}}{a^2}=1.$$

(*H. C. S.* 89—111; *Hust.* Prop. 8—12; *Biot*, 133—60.)

(15.) *The ellipse referred to the focus.* Let S, H, be the foci, then

$$SP=a-ex,\qquad HP=a+ex.$$

$$SP+HP=2a=Aa.$$

The polar equation to the ellipse, the focus being the pole, is

$$u=a\frac{1-e^2}{1+e\cos\theta}.$$

$$SP=\frac{BC^2}{AC-SC.\cos PSN}.$$

$$\frac{1}{SP}+\frac{1}{Sp}=\frac{2}{SL}.$$

If the centre is the pole, the equation is

$$u=a\,.\left.\overline{\frac{1-e^2}{1-(e\cos\theta)^2}}\right|^{\frac{1}{2}}.$$

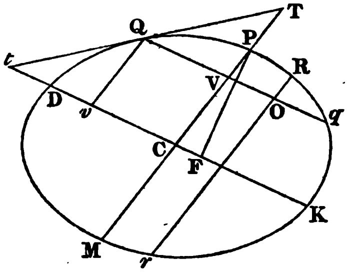
The angle $SPT=HPt$.

If SY, HZ be drawn perpendicular to the tangent, the points Y, Z are always in the circumference of a circle described on Aa.

$PI=AC$.

$SY.HZ=BC^2$. (*H. C. S.* 112–22; *Hust.* Prop. 1–7.)

(16.) *The ellipse referred to any conjugate diameters.*



If CD is conjugate to CP, conversely CP is conjugate to CD.

A tangent at P is parallel to CD.

The axes are the only conjugate diameters that are perpendicular to each other.

The equation to an ellipse referred to the centre and any conjugate diameters a_1, b_1 is

$$\frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}=1.$$

If the extremity of the diameter is the origin, then

$$y^2 = \frac{b_1^{\,2}}{a_1^{\,2}}(2a_1x - x^2).$$

$$PV.Vp : QV^2 :: CP^2 : CD^2.$$

$$CV.CT = CP^2;\ \text{and}\ Cv.Ct = CD^2.$$

If any chords Qq, Rr, parallel to CD, CP, respectively, intersect each other in O, then

$$QO.Oq : RO.Or :: CD^2 : CP^2.$$

If a_1, b_1, are any conjugate diameters,

$$a_1^{\,2} + b_1^{\,2} = a^2 + b^2.$$

The area of all circumscribing parallelograms is constant, and

$$= 4CD.PF = 4AC.BC.$$

If $a_1 = b_1$, then $a_1 = \overline{\tfrac{1}{2}(a^2+b^2)}|^{\frac{1}{2}}$;

$$\sin a_1,b_1 = \frac{2ab}{a^2+b^2}:$$

$$\tan a_1,a = \frac{b}{a}.$$

$SP.HP = PT.Pt = CD^2$. (*See Fig.* p. 163.)

$AM.am = BC^2$.

Let Pp be the diameter drawn through any point P, whose co-ordinates are x_1, y_1, and PQ, Qp the supplemental chords: the equation to PQ being

$$y - y_1 = \alpha(x - x_1),$$

the equation to Qp is

$$y + y_1 = -\frac{b_1^{\,2}}{a_1^{\,2}\alpha}(x + x_1).$$

If the chords be drawn from the extremities of the axis major, their equations are

$$y = \alpha(x + a);$$

$$y = -\frac{b_1^{\,2}}{a_1^{\,2}\alpha}(x - a).$$

If two diameters be drawn parallel to any two supplemental chords, they are conjugate to each other.

$$\tan APa = -\frac{2ab^2}{y_1(a^2-b^2)}.$$

(*H. C. S.* 127—64; *Hust.* Prop. 13—8; *Biot*, 161—90.)

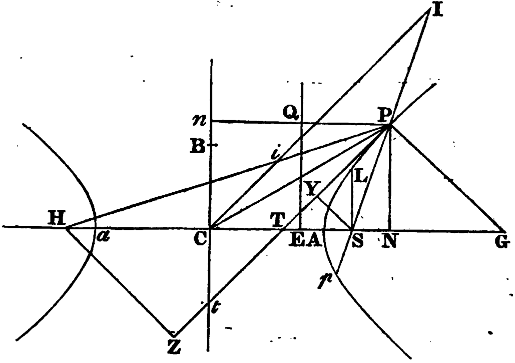
Radius of curvature $= \dfrac{CD^2}{PF}$;

Chord of curvature through $C = \dfrac{2CD^2}{CP}$;

Chord through the focus $= \dfrac{2CD^2}{AC}$. (*Hust.* Prop. 19.)

The Hyperbola.

(17.) *The Hyperbola referred to its axes.*



$SP : PQ :: e : 1$; e being > 1.*

If the curve is referred to the axis, and a tangent at the vertex, the equation is

$$y^2 = \frac{b^2}{a^2}(2ax + x^2).$$

$AN . Na : NP^2 :: AC^2 : BC^2$.

If the origin is at the centre, the equation is

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2); \quad \text{or } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

---

* The lines in the above figure are analogous to those in the ellipse, on which the same letters are placed.

The equation to the *equilateral* hyperbola is

$$x^2 - y^2 = a^2.$$

$$SL = \pm \frac{b^2}{a} = \frac{BC^2}{AC}.$$

The intersections of the hyperbola with the straight line,

$$y = \alpha x + \beta$$

may be found by the solution of the equation

$$y^2 + \frac{2b^2\beta}{a^2\alpha^2 - b^2}y + \frac{b^2(a^2\alpha^2 - \beta^2)}{a^2\alpha^2 - b^2} = 0,$$

which has only two roots; a straight line therefore cannot cut an hyperbola in more than two points.

The equation to a tangent at the point (x_1, y_1) is

$$y - y_1 = \frac{b^2}{a^2} \cdot \frac{x_1}{y_1}(x - x_1),$$

$$\text{or } \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

$CN . CT = CA^2.$

$Cn . Ct \ = CB^2.$

The subtangent, $CT = \dfrac{x_1^2 - a^2}{x_1}.$

The equation to the normal is

$$y - y_1 = -\frac{a^2}{b^2} \cdot \frac{y_1}{x_1}(x - x_1).$$

The subnormal, $NG = \dfrac{b^2}{a^2}x_1 = \dfrac{BC^2}{AC^2}CN.$

To draw a tangent from a point (x_2, y_2) without the hyperbola: the point (x_1, y_1) at which the tangent meets the curve must be determined from the equations

$$\frac{x_2x_1}{a^2} - \frac{y_2y_1}{b^2} = 1,$$

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1.$$

(*H. C. S.* 172—94; *Hust.* Prop. 7, 8; *Biot,* 224—44.)

(18.) *The hyperbola referred to the focus.*

$SP = ex - a$; and $HP = ex + a$.

$HP \sim SP = 2AC$.

The polar equation, the focus S being the pole, is

$$u = a\frac{e^2 - 1}{1 + e\cos\theta}.$$

If H be the pole, then

$$u = -a\frac{e^2 - 1}{1 - e\cos\theta}.$$

$$SP = \frac{BC^2}{AC - SC.\cos PSN}.$$

$$\frac{1}{SP} + \frac{1}{Sp} + \frac{2}{SL}.$$

$SP.Sp = \frac{1}{2}a(e^2 - 1)(SP + Sp)$.

If the centre is the pole, the equation is

$$u = a.\left.\frac{e^2 - 1}{1 - (e\cos\theta)^2}\right|^{\frac{1}{2}}$$

$\angle HPT = \angle SPT$.

If SY and HZ be drawn perpendicular to PT, the locus of the points Y and Z is a circle described on Aa.

$PI = Pi = AC$.

$SY.HZ = BC^2$.

(*H. C. S.* 195—205; *Biot*, 259—63; *Hust.* Prop. 1—6.)

(19.) *The hyperbola referred to any system of conjugate diameters.*

The locus of the points of bisection of all parallel chords is a diameter.

The co-ordinates of the points of intersection of any diameter $y = \alpha x$ with the curve are

$$x = \pm\frac{ab}{(b^2 - a^2\alpha^2)^{\frac{1}{2}}}, \qquad y = \pm\frac{ab\alpha}{(b^2 - a^2\alpha^2)^{\frac{1}{2}}}.$$

In order that the diameter may meet the curve, α must be $< \frac{b}{a}$, and $> -\frac{b}{a}$.

If the diameter a_1 is conjugate to b_1, b_1 is conjugate to a_1.

The equation to the diameter passing through (x_1, y_1) is

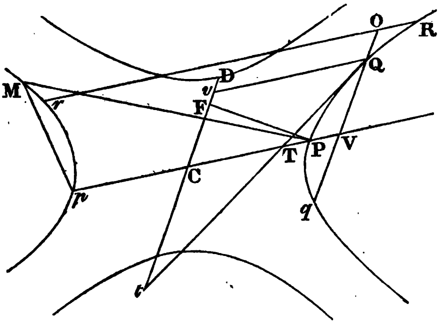
$$y = \frac{y_1}{x_1} x.$$

the equation to the conjugate diameter is

$$y = \frac{b^2}{a^2} \cdot \frac{x_1}{y_1} \cdot x.$$

The equation to an hyperbola referred to the centre, and two conjugate diameters a_1, b_1, is,

$$\frac{x^2}{a_1^{\,2}} - \frac{y^2}{b_1^{\,2}} = 1.$$



If P, the extremity of a_1, is the origin, the equation is

$$y^2 = \frac{b_1^{\,2}}{a_1^{\,2}} (x^2 + 2a_1 x).$$

$$PV.Vp : QV^2 :: CP^2 : CD^2.$$

$$CV.CT = CP^2; \quad Cv.Ct = CD^2.$$

If from the several points of a straight line given in position, pairs of tangents be drawn to an hyperbola, the lines joining the corresponding points of contact will all pass through the same point.

$QO \cdot Oq : RO \cdot Or :: CD^2 : CP^2$.

$a_1^2 - b_1^2 = a^2 - b^2$.

If tangents to the conjugate hyperbolas be drawn at the extremities of the conjugate diameters, the area of the parallelogram is constant, and $= 4CD \cdot PF = 4AC \cdot BC$.

$SP \cdot HP = CD^2$.

$PT \cdot Pt = CD^2$.

If the equation to PM is

$$y - y_1 = a(x - x_1),$$

the equation to mp is

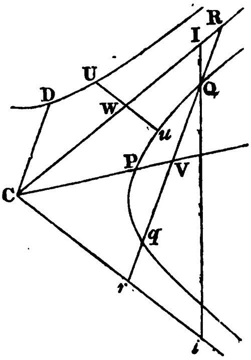
$$y + y_1 = \frac{b_1^2}{a_1^2 a}(x + x_1).$$

If diameters be drawn parallel to any two supplemental chords, they are conjugate.

(*H. C. S.* 206—45; *Biot*, 245—58; *Hust.* Prop. 15—8.)

(20.) *The asymptotes of the hyperbola.* If the origin is at the centre, and a, b are any system of conjugate diameters, the equation to the asymptotes is

$$y^2 = \frac{b^2}{a^2}x^2.$$



$RQ = rq$.

$QI \cdot Qi = BC^2$.

$QR \cdot Qr = CD^2$.

If the asymptotes be the axes, and the centre the origin, the equation to the hyperbola is

$$xy = \tfrac{1}{4}(a^2 + b^2).$$

The equation to a tangent at the point (x_1,y_1) is

$$xy_1 + x_1y = \tfrac{1}{2}(a^2 + b^2).$$

The area of a triangle contained between these axes and a tangent is constant, and $= \frac{1}{2}(a^2 + b^2)\sin x,y$.

Let (x_1,y_1) be any point in the curve, and a_1, b_1 the conjugate diameters to that point, then

$$a_1 = \pm 2\,(x_1 \,.\, y_1)^{\frac{1}{2}} \cos x,y\,; \qquad b_1 = \pm 2\,(x_1 \,.\, y_1)^{\frac{1}{2}} \sin x,y.$$

If UWu is parallel to Ci, then $UW = Wu$.

(*H. C. S.* 245—59; *Biot*, 250—8; *Hust.* Prop. 11—5.)

Discussion of Lines of the Second Order.

(21.) The general form of an equation of the second degree containing two unknown quantities is

$$ay^2 + bxy + cx^2 + dy + ex + f = 0.$$

From the solution of this equation

$$y = -\frac{bx+d}{2a} \pm \frac{1}{2a}\,.\,\overline{(b^2 - 4ac)x^2 + 2(bd - 2ae)x + d^2 - 4af}\Big|^{\frac{1}{2}};$$

$$x = -\frac{by+e}{2c} \pm \frac{1}{2c}\,.\,\overline{(b^2 - 4ac)y^2 + 2(be - 2cd)y + e^2 - 4cf}\Big|^{\frac{1}{2}}.$$

which values of x and y may be thus represented:

$$y = -\frac{bx+d}{2a} \pm \frac{1}{2a}\,.\,\overline{Ax^2 + 2Bx + C}\Big|^{\frac{1}{2}};$$

$$x = -\frac{by+e}{2c} \pm \frac{1}{2c}\,.\,\overline{Ay^2 + 2B_1y + C_1}\Big|^{\frac{1}{2}}.$$

A straight line cannot cut a line of the second order in more than two points.

The locus of the middle points of any number of parallel chords is a diameter.

If the chords drawn parallel to any diameter are bisected by another diameter, then the chords parallel to the latter, will be bisected by the former.

The co-ordinates of the centre of the curve are

$$x = \frac{2ae - bd}{b^2 - 4ac}\,; \qquad y = \frac{2cd - be}{b^2 - 4ac}.$$

If $b^2-4ac<0$, the curve is an ellipse;

$b^2-4ac>0$, ... an hyperbola;

$b^2-4ac=0$, ... a parabola.

If $b=0$, a and c must have the same sign in the ellipse, and different signs in the hyperbola.

If the curve is an ellipse or hyperbola, and the axes parallel to any system of conjugate diameters, the form of the equation is

$$ay^2+cx^2+dy+ex+f=0.$$

If the origin is at the centre of the curve, the equation becomes

$$ay^2+bxy+cx^2+f=0:$$

and if the curve is also referred to conjugate diameters, then

$$ay^2+cx^2+f=0.$$

If the axes are parallel to the asymptotes of an hyperbola, the form of the equation is

$$bxy+dy+ex+f=0.$$

If $e^2>4cf$, the curve intersects the axis of x in two points;

$e^2=4cf$, ... touches the axis of x;

$e^2<4cf$, ... does not meet the axis of x;

The same conditions exist with regard to the axis of y, according as $d^2>$, $=$, or $<4af$.

If $B^2>AC$, the curve intersects the diameter, $y=-\dfrac{bx+d}{2a}$;

$B^2=AC$, ... touches that diameter;

$B^2<AC$, ... does not meet it.

(22.) *Determination of the species, and their varieties.*

[1] If $A<0$, the curve is an ellipse.

If the roots of the equation

$$x^2+\frac{2B}{A}x+\frac{C}{A}=0, \qquad (a)$$

are equal, the ellipse is reduced to a point.

If the roots are impossible, the ellipse becomes an imaginary line.

If $a=c$, and $b=0$, the ellipse becomes a circle.

[2] If $A > 0$, the curve is an hyperbola.

If the roots of the equation (a) are possible, the diameter,

$$y = -\frac{bx + d}{2c}$$

cuts the curve; if impossible, that diameter does not meet it.

If the roots are equal, the hyperbola becomes two intersecting straight lines.

If $a = -c$, and $b = 0$, the hyperbola is equilateral.

[3] If $A = 0$, the curve is a parabola.

If B also $= 0$, and

$C > 0$, the equation represents two parallel straight lines,

$C = 0$, ... one straight line,

$C < 0$, ... an imaginary line.

(*H. A. G.* 81–89, 184–213; *G. G. A.* Ch. iv; *Biot*, 264–304.)

SUMMARY OF EQUATIONS.

(23.) *Rectilinear equations.*

[1] Let the curve be referred to its centre, and principal diameter, then

in the ellipse, $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$;

. . . . hyperbola, $y^2 = \frac{b^2}{a^2}(x^2 - a^2)$;

. . . . circle, $y^2 = a^2 - x^2$;

. . . . equilateral hyperbola, $y^2 = x^2 - a^2$.

[2] Let the curve be referred to the principal diameter, and a tangent at the vertex, then

in the ellipse, $y^2 = \frac{b^2}{a^2}(2ax - x^2)$;

. . . . hyperbola, $y^2 = \frac{b^2}{a^2}(2ax + x^2)$;

. . . . parabola, $y^2 = 4mx$;

. . . . circle, $y^2 = 2ax - x^2$;

. . . . equilateral hyperbola, $y^2 = 2ax + x^2$.

The general form of the equation is $y^2 = mx + nx^2$.

In the Index

The same equations subsist, if a and b are any system of conjugate diameters to which the curve is referred.

(24.) *Polar equations.*

[1] Let the curve be referred to the radius vector, and angle contained between it, and the principal diameter, then

1. The centre being the pole;

$$\text{in the ellipse,}\quad u^2 = \frac{a^2(1-e^2)}{1-(e\,.\cos\theta)^2},$$

$$\ldots\ldots\text{hyperbola,}\quad u^2 = \frac{a^2(e^2-1)}{(e\,.\cos\theta)^2-1}:$$

2. The focus being the pole;

$$\text{in the ellipse,}\quad u = \frac{a(1-e^2)}{1+e\,.\cos\theta},$$

$$\ldots\ldots\text{hyperbola,}\quad u = \frac{a(e^2-1)}{1+e\cos\theta},$$

$$\ldots\ldots\text{parabola,}\quad u = \frac{2m}{1+\cos\theta}.$$

[2] Equations between the radius vector, and a perpendicular on the tangent from the pole.

1. The centre being the pole:

$$\text{in the ellipse,}\quad p^2 = \frac{a^2b^2}{a^2+b^2-u^2},$$

$$\ldots\ldots\text{hyperbola,}\quad p^2 = \frac{a^2b^2}{u^2-(a^2-b^2)},$$

$$\ldots\ldots\text{equilateral hyperbola,}\quad p = \frac{a^2}{u}:$$

2. The focus being the pole;

$$\text{in the ellipse,}\quad p^2 = \frac{b^2u}{2a-u},$$

$$\ldots\ldots\text{hyperbola,}\quad p^2 = \frac{b^2u}{2a+u},$$

$$\ldots\ldots\text{parabola,}\quad p^2 = mu.$$ (*H. A. G.* 183.)

Analytical Geometry of three Dimensions.

(25.) *The straight line.* The equations to a straight line referred to three rectangular co-ordinates, are

$$x = az + \alpha, \qquad y = bz + \beta:$$

which are the equations to its projections on the planes of xz, yz respectively.

The equations to a line passing through the point (x_1, y_1, z_1), are

$$x - x_1 = a(z - z_1), \qquad y - y_1 = b(z - z_1).$$

The equations to a line passing through two points (x_1, y_1, z_1), (x_2, y_2, z_2), are

$$x - x_1 = \frac{x_2 - x_1}{z_2 - z_1}(z - z_1), \qquad y - y_1 = \frac{y_2 - y_1}{z_2 - z_1}(z - z_1).$$

In order that the two lines

$$\begin{cases} x = a_1 z + \alpha_1, \\ y = b_1 z + \beta_1; \end{cases} \qquad \begin{cases} x = a_2 z + \alpha_2, \\ y = b_2 z + \beta_2, \end{cases}$$

may intersect each other, it is necessary that

$$\frac{\alpha_1 - \alpha_2}{a_1 - a_2} = \frac{\beta_1 - \beta_2}{b_1 - b_2}.$$

The co-ordinates of the point of intersection are

$$x = \frac{a_1 \alpha_2 - a_2 \alpha_1}{a_1 - a_2}, \quad y = \frac{b_1 \beta_2 - b_2 \beta_1}{a_1 - a_2}, \quad z = \frac{\alpha_2 - \alpha_1}{a_1 - a_2}.$$

The distance (d_1) of a point (x_1, y_1, z_1) from the origin;

$$d_1 = (x_1^2 + y_1^2 + z_1^2)^{\frac{1}{2}} = z(1 + a^2 + b^2)^{\frac{1}{2}};$$

if $x = az$, $y = bz$ are the equations to the line passing through the given point, and the origin.

The distance (D) between two points (x_1, y_1, z_1), (x_2, y_2, z_2);

$$D = \overline{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}\Big|^{\frac{1}{2}}:$$

$$= \overline{d_1^2 + d_2^2 - 2(x_1 x_2 + y_1 y_2 + z_1 z_2)}\Big|^{\frac{1}{2}},$$

d_2 being the distance of (x_2, y_2, z_2) from the origin.

The inclination of a straight line l to each of the axes:

$$\cos l{,}x = \frac{\pm a}{(1+a^2+b^2)^{\frac{1}{2}}},$$

$$\cos l{,}y = \frac{\pm b}{(1+a^2+b^2)^{\frac{1}{2}}},$$

$$\cos l{,}z = \frac{\pm 1}{(1+a^2+b^2)^{\frac{1}{2}}},$$

$$(\cos l{,}x)^2 + (\cos l{,}y)^2 + (\cos l{,}z)^2 = 1.$$

If $l{,}xy$, &c. denote the angles which l makes with the planes xy, &c. then

$$(\sin l{,}xy)^2 + (\sin l{,}xz)^2 + (\sin l{,}yz)^2 = 1.$$

The mutual inclination of two lines, l_1, l_2:

$$\cos l_1{,}l_2 = \cos l_1{,}x . \cos l_2{,}x + \cos l_1{,}y . \cos l_2{,}y + \cos l_1{,}z . \cos l_2{,}z.$$

$$= \pm \frac{1 + a_1a_2 + b_1b_2}{\overline{(1+a_1{}^2+b_1{}^2)(1+a_2{}^2+b_2{}^2)}|^{\frac{1}{2}}}.$$

The equations to a line in terms of the angles it makes with the axes are

$$x = \frac{\cos l{,}x}{\cos l{,}z} z + a, \qquad y = \frac{\cos l{,}y}{\cos l{,}z} z + \beta.$$

(*G. G. A.* Ch. xvi; *H. A. G.* 229—51; *Biot,* 54—69.)

(26.) *The plane.* The most general form of the equation to a plane is

$$Ax + By + Cz + D = 0, \text{ or}$$

$$z = A_1x + B_1y + C_1.$$

The equations to the *traces* of the given plane on the planes of xy, xz, yz, respectively are

$$Ax + By + D = 0, \qquad Ax + Cz + D = 0, \qquad By + Cz + D = 0.$$

If a, b, c, are the distances from the origin at which the plane cuts the axes of x, y, z, respectively, the equation becomes

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

If ρ is the perpendicular on the plane from the origin, then

$$\rho = x \cos \rho{,}x + y \cos \rho{,}y + z \cos \rho{,}z.$$

The equation to a plane passing through the point (x_1, y_1, z_1), and parallel to the plane $Ax + By + Cz + D = 0$, is

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0.$$

The co-ordinates of the intersection of a straight line

$$x = az + \alpha, \qquad y = bz + \beta$$

and the plane $Ax + By + Cz + D = 0$, are

$$x = \frac{\alpha(Aa + Bb + C) - a(A\alpha + B\beta + D)}{Aa + Bb + C},$$

$$y = \frac{\beta(Aa + Bb + C) - b(A\alpha + B\beta + D)}{Aa + Bb + C},$$

$$z = -\frac{A\alpha + B\beta + D}{Aa + Bb + C}.$$

When the line and plane are parallel, then

$$Aa + Bb + C = 0.$$

The equations to a perpendicular p from the point (x_1, y_1, z_1) on the plane, are

$$x - x_1 = \frac{A}{C}(z - z_1), \qquad y - y_1 = \frac{B}{C}(z - z_1).$$

$$p = \pm \frac{Ax_1 + By_1 + Cz_1 + D}{(A^2 + B^2 + C^2)^{\frac{1}{2}}}.$$

The equation to a plane drawn through a point (x_1, y_1, z_1) perpendicular to a given line, is

$$a(x - x_1) + b(y - y_1) + z - z_1 = 0.$$

The angle contained between the line l, and the plane P;

$$\sin l, P = \frac{Aa + Bb + C}{\overline{(1 + a^2 + b^2)(A^2 + B^2 + C^2)}|^{\frac{1}{2}}}.$$

(27.) The inclination of a plane P to the co-ordinate planes:

$$\cos P, xy = \frac{\pm C}{(A^2 + B^2 + C^2)^{\frac{1}{2}}},$$

$$\cos P, xz = \frac{\pm B}{(A^2 + B^2 + C^2)^{\frac{1}{2}}},$$

$$\cos P, yz = \frac{\pm A}{(A^2 + B^2 + C^2)^{\frac{1}{2}}}.$$

$$(\cos P, xy)^2 + (\cos P, xz)^2 + (\cos P, yz)^2 = 1.$$

The mutual inclination of two planes P_1, P_2;

$$\cos P_1, P_2 = \cos P_1, xy \,.\, \cos P_2, xy + \cos P_1, xz \,.\, \cos P_2, xz$$
$$+ \cos P_1, yz \,.\, \cos P_2, yz.$$
$$= \frac{A_1 A_2 + B_1 B_2 + C_1 C_2}{\{(A_1^2 + B_1^2 + C_1^2)(A_2^2 + B_2^2 + C_2^2)\}^{\frac{1}{2}}}.$$

If the planes are parallel, then

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}.$$

(*H. A. G.* 252—74; *Biot*, 70—85; *G. G. A.* Ch. xvii.)

THE ORTHOGONAL PROJECTION OF PLANE FIGURES.

(28.) Let A represent the area of a plane figure, A_{xy}, A_{xz}, A_{yz}, the areas of its projections on the planes xy, xz, yz respectively, then
$$A_{xy}^2 + A_{xz}^2 + A_{yz}^2 = A^2.$$

If A, A_{yz}, be projected on the plane $y_1 z_1$ of some new system, then

$$A_{y_1 z_1} = A_{yz} \cos x_1, x + A_{xz} \cos x_1, y + A_{xy} \cos x_1, z.$$

If A_1, A_2, A_3, &c. are any areas in different planes,

X, Y, Z, the sums of their projections on yz, xz, xy;

X_1, Y_1, Z_1, ... $y_1 z_1$, $x_1 z_1$, $x_1 y_1$;

then
$$X_1 = X \cos x_1, x + Y \cos x_1, y + Z \cos x_1, z,$$
$$Y_1 = X \cos y_1, x + Y \cos y_1, y + Z \cos y_1, z,$$
$$Z_1 = X \cos z_1, x + Y \cos z_1, y + Z \cos z_1, z.$$
$$X^2 + Y^2 + Z^2 = X_1^2 + Y_1^2 + Z_1^2.$$

If the triangle formed by joining any three points in space be represented by a, and the perpendicular on the plane from the origin, by ρ, the equation to the plane passing through the three points is

$$a_{yz} \,.\, x + a_{xz} \,.\, y + a_{xy} \,.\, z = a\rho.$$

(*H. A. G.* 275—90; *G. G. A.* p. 327—32.)

OBLIQUE CO-ORDINATES.

(29.) The equations to the line l referred to oblique co-ordinates are

$$x = \frac{\sin l_{xz}, z}{\sin l_{xz}, xy} z + a, \qquad y = \frac{\sin l_{yz}, z}{\sin l_{yz}, xy} z + \beta.$$

The distance of the point (x,y,z) from the origin

$$= x^2 + y^2 + z^2 + 2(xy \cos x,y + xz \cos x,z + yz \cos y,z).$$

The distance between the two points $(x_1,y_1,z_1), (x_2,y_2,z_2)$

$$\begin{aligned} &= (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 \\ &+ 2\{(x_1 - x_2)(y_1 - y_2) \cos x,y + (x_1 - x_2)(z_1 - z_2) \cos x,z \\ &+ (y_1 - y_2)(z_1 - z_2) \cos y,z\}. \end{aligned}$$

The inclination of two lines

$$\left\{\begin{aligned} x &= a_1 z + a_1 \\ y &= b_1 z + \beta_1 \end{aligned}\right. \quad (l_1), \qquad \left\{\begin{aligned} x &= a_2 z + a_2 \\ y &= b_2 z + \beta_1 \end{aligned}\right. \quad (l_2):$$

$$\begin{aligned} \cos l_1,l_2 &= \frac{\sin l_2,yz}{\sin x,yz} \cos l_1,x + \frac{\sin l_2,xz}{\sin y,xz} \cos l_1,y + \frac{\sin l_2,xy}{\sin z,xy} \cos l_1,z, \\ &= \frac{\sin l_1,yz}{\sin x,yz} \cos l_2,x + \frac{\sin l_1,xz}{\sin y,xz} \cos l_2,y + \frac{\sin l_1,xy}{\sin z,xy} \cos l_2,z; \end{aligned}$$

$$= \frac{a_1 a_2 + b_1 b_2 + 1 + (a_1 b_2 + a_2 b_1)\cos x,y + (a_1 + a_2)\cos x,z + (b_1 + b_2)\cos y,z}{\left[\begin{aligned} &\{a_1^2 + b_1^2 + 1 + 2(a_1 b_1 \cos x,y + a_1 \cos x,z + b_1 \cos y,z)\} \\ \times &\{a_2^2 + b_2^2 + 1 + 2(a_2 b_2 \cos x,y + a_2 \cos x,z + b_2 \cos y,z)\} \end{aligned}\right]^{\frac{1}{2}}}.$$

The inclination of a plane P to each of the co-ordinate planes: let ρ be the perpendicular from the origin, then the equation to the plane is

$$\begin{aligned} \rho &= x \cos \rho,x + y \cos \rho,y + z \cos \rho,z, \\ &= Ax + By + Cz. \end{aligned}$$

$$\cos P,yz = \frac{1}{\sin x,yz} A + \frac{\cos yz,xz}{\sin y,xz} B + \frac{\cos yz,xy}{\sin z,xy} C,$$

$$\cos P,xz = \frac{1}{\sin y,xz} B + \frac{\cos xz,yz}{\sin x,yz} A + \frac{\cos xz,xy}{\sin z,xy} C,$$

$$\cos P,xy = \frac{1}{\sin z,xy} C + \frac{\cos xy,yz}{\sin x,yz} A + \frac{\cos xy,xz}{\sin y,xz} B.$$

$$\frac{\cos P,yz}{\sin x,yz}A+\frac{\cos P,xz}{\sin y,xz}B+\frac{\cos P,xy}{\sin z,xy}C=1.$$

Let $\angle\, xy,xz=\alpha,\quad \angle\, xy,yz=\beta,\quad \angle\, xz,yz=\gamma;$
$\sin x,yz=d,\quad \sin y,xz=e,\quad \sin z,xy=f$: then

$$\frac{A^2}{d^2}+\frac{B^2}{e^2}+\frac{C^2}{f^2}-2\left(\frac{AB}{de}\cos\gamma+\frac{AC}{df}\cos\beta+\frac{BC}{ef}\cos\alpha\right)=1.$$

The inclination of two planes

$$A_1x+B_1y+C_1z=\rho_1,\qquad (P_1)$$
$$A_2x+B_2y+C_2z=\rho_2;\qquad (P_2)$$

$$-\cos P_1,P_2=$$

$$\frac{\dfrac{A_1A_2}{d^2}+\dfrac{B_1B_2}{e^2}+\dfrac{C_1C_2}{f^2}-\dfrac{A_1B_2+A_2B_1}{de}\cos\gamma-\dfrac{A_1C_2+A_2C_1}{df}\cos\beta-\dfrac{B_1C_2+B_2C_1}{ef}\cos\gamma}{\left[\left\{\dfrac{A_1^{\,2}}{d^2}+\dfrac{B_1^{\,2}}{e^2}+\dfrac{C_1^{\,2}}{f^2}-2\left(\dfrac{A_1B_1}{de}\cos\gamma+\dfrac{A_1C_1}{df}\cos\beta+\dfrac{B_1C_1}{ef}\cos\alpha\right)\right\}\times\left\{\dfrac{A_2^{\,2}}{d^2}+\dfrac{B_2^{\,2}}{e^2}+\dfrac{C_2^{\,2}}{f^2}-2\left(\dfrac{A_2B_2}{de}\cos\gamma+\dfrac{A_2C_2}{df}\cos\beta+\dfrac{B_2C_2}{ef}\cos\alpha\right)\right\}\right]^{\frac{1}{2}}}.$$

(*Whewell, Camb. Phil. Trans.* V. 2, Pt. 1; *H. A. G.* Ch. ix, x.)

THE TRANSFORMATION OF CO-ORDINATES.

(30.) Let x, y, z, be the original, x_1, y_1, z_1, the new co-ordinates, then

$$x=\frac{1}{\sin x,yz}(x_1\sin x_1,yz+y_1\sin y_1,yz+z_1\sin z_1,yz),$$
$$y=\frac{1}{\sin y,xz}(x_1\sin x_1,xz+y_1\sin y_1,xz+z_1\sin z_1,xz),$$
$$z=\frac{1}{\sin z,xy}(x_1\sin x_1,xy+y_1\sin y_1,xy+z_1\sin z_1,xy).$$

If the primitive axes are rectangular, and the new ones oblique, then

$$\left.\begin{aligned}x&=x_1\cos x_1,x+y_1\cos y_1,x+z_1\cos z_1,x,\\ y&=x_1\cos x_1,y+y_1\cos y_1,y+z_1\cos z_1,y,\\ z&=x_1\cos x_1,z+y_1\cos y_1,z+z_1\cos z_1,z:\end{aligned}\right.\qquad(1)$$

$$(\cos x_1,x)^2 + (\cos x_1,y)^2 + (\cos x_1,z)^2 = 1,$$
$$(\cos y_1,x)^2 + (\cos y_1,y)^2 + (\cos y_1,z)^2 = 1,$$
$$(\cos z_1,x)^2 + (\cos z_1,y)^2 + (\cos z_1,z)^2 = 1.$$

If both systems are rectangular, then

$$\cos x_1,x \,.\, \cos y_1,x + \cos x_1,y \,.\, \cos y_1,y + \cos x_1,z \,.\, \cos y_1,z = 0,$$
$$\cos x_1,x \,.\, \cos z_1,x + \cos x_1,y \,.\, \cos z_1,y + \cos x_1,z \,.\, \cos z_1,z = 0,$$
$$\cos y_1,x \,.\, \cos z_1,x + \cos y_1,y \,.\, \cos z_1,y + \cos y_1,z \,.\, \cos z_1,z = 0.$$

(31.) Let t represent the *trace* of the plane xy on x_1y_1, then if both systems are rectangular, the position of the new axes may be determined from the three angles

$$t,x = \phi, \quad t,x_1 = \psi, \quad xy,x_1y_1 = \chi.$$

The formulæ (1) become respectively

$$x = x_1 (\cos\phi.\cos\psi + \sin\phi.\cos\psi.\cos\chi)$$
$$+ y_1 (-\cos\phi.\sin\psi + \sin\phi.\cos\psi.\cos\chi) - z_1 \sin\phi.\sin\chi,$$
$$y = x_1 (\sin\phi.\cos\psi - \cos\phi.\sin\psi.\cos\chi)$$
$$- y_1 (\sin\phi.\sin\psi + \cos\phi.\cos\psi.\cos\chi) + z_1 \cos\phi.\sin\chi,$$
$$z = x_1 \sin\psi.\sin\chi + y_1 \cos\psi.\sin\chi + z_1 \cos\chi.$$

If t be supposed to coincide with x_1, these formulæ become

$$x = x_1 \cos\phi + y_1 \sin\phi.\cos\chi,$$
$$y = x_1 \sin\phi - y_1 \cos\phi.\cos\chi + z_1 \cos\phi.\sin\chi, \qquad (2)$$
$$z = y_1 \sin\chi + z_1 \cos\chi.$$

If the origin alone be changed, then

$$x = x_1 + a, \qquad y = y_1 + b, \qquad z = z_1 + c.$$

If the position of the origin, as well as the direction of the axes be changed, then a, b, c, the co-ordinates of the new origin must be respectively added to the above values of x, y, and z.

(32.) The equation to any surface being given, to find the equation to the curve formed by its intersection with a plane; let the equation to the surface be

$$f(x,y,z) = 0,$$

and the plane of x_1y_1 the intersecting plane, then the trace of

x_1y_1 on xy being assumed the axis of x, and a line perpendicular to x_1 in the plane x_1y_1 the axis of y_1, (2) become

$$x = x_1 \cos\phi + y_1 \sin\phi . \cos\chi; \qquad (3)$$
$$y = x_1 \sin\phi - y_1 \cos\phi . \cos\chi,$$
$$z = y_1 \sin\chi.$$

These values of x, y, z being substituted in the equation to the surface will give the equation to the curve required.

(33.) *Transformation of rectangular to polar co-ordinates.*

Let u be the radius vector, u_{xy} its projection on the plane xy, and a, b, c the co-ordinates of the pole, then

$$x = u \cos u_{xy}, u . \cos u_{xy}, x + a,$$
$$y = u \cos u_{xy}, u . \sin u_{xy}, x + b,$$
$$z = u \sin u_{xy}, u + c.$$

The radius vector must always be considered positive.

(*H. A G.* Ch. xi; *G. G. A.* Ch. xviii; *Biot*, 97—103.)

THE SPHERE.

(34.) Let α, β, γ, be the co-ordinates of the centre; and r the radius; the most general form of the equation is

$$2\{(x-\alpha)(y-\beta)\cos x,y + (x-\alpha)(z-\gamma)\cos x,z + (y-\beta)(z-\gamma)\cos y,z\}$$
$$+ (x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = r^2. \qquad (1)$$

If the origin is at the centre, the equation becomes

$$x^2 + y^2 + z^2 + 2(xy \cos x,y + xz \cos x,z + yz \cos y,z) = r^2.$$

If the axes are rectangular, (1) becomes

$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = r^2.$$

If the origin is on the surface of the sphere, then

$$x^2 + y^2 + z^2 - 2(\alpha x + \beta y + \gamma z) = 0.$$

If the origin is at the centre, then

$$x^2 + y^2 + z^2 = r^2.$$

The equation to a plane touching the sphere at the point (x_1,y_1,z_1) is

$$(x-\alpha)(x_1-\alpha) + (y-\beta)(y_1-\beta) + (z-\gamma)(z_1-\gamma) = r^2.$$

If the origin is at the centre, this equation becomes

$$xx_1 + yy_1 + zz_1 = r^2. \qquad (H. A. G. 323—9.)$$

THE CYLINDER.

(35.) The general equation to a cylindrical surface, the *directrix* of which is in the plane xy, is

$$f(x - az,\ y - bz) = 0.$$

The equation to an oblique cylinder on a circular base, in the plane xy, the origin being ~~in the circumference of the base,~~ is

$$(x - az)^2 + (y - bz)^2 = 2r\,(x - az).$$

Every section of this cylinder made by a plane inclined to its axis is an ellipse, or a circle. (*H. A. G.* 331—4.)

THE CONE.

(36.) Let α, β, γ, be the co-ordinates of the vertex, and let the directrix be in the plane xy, the general equation to a conical surface is

$$f\left(\frac{x-\alpha}{z-\gamma},\ \frac{y-\beta}{z-\gamma}\right) = 0.$$

The equation to a right cone on a circular base in the plane xy is

$$\left.\overline{\frac{x-\alpha}{z-\gamma}}\right|^2 + \left.\overline{\frac{y-\beta}{z-\gamma}}\right|^2 = \frac{r^2}{\gamma^2}.$$

If the origin is at the centre of the base, the equation becomes

$$x^2 + y^2 = \frac{r^2}{\gamma^2}(z - \gamma)^2.$$

The equation to a right cone, of which the base is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left.\overline{\frac{z-\gamma}{\gamma}}\right|^2.$$

The equation to an oblique cone on a circular base, the origin being in the circumference of the base, is

$$(\alpha z - \gamma x)^2 + (\beta z - \gamma y)^2 = 2r\,(z - \gamma)\,(\alpha z - \gamma x).$$

If the origin is at the centre of the base, the equation becomes

$$(\alpha z - \gamma x)^2 + (\beta z - \gamma y)^2 = r^2\,(z - \gamma)^2.$$

(37.) The equation to the intersection of the cone

$$x^2+y^2=\frac{r^2}{\gamma^2}(z-\gamma)^2$$

with a plane passing through the origin, perpendicular to the plane xz, and inclined to the plane xy at an angle χ, is

$$\{(\gamma\cos\chi)^2-(r\sin\chi)^2\}\,y_1^2+\gamma^2x_1^2+2\gamma r^2\sin\chi\cdot y_1-r^2\gamma^2=0.$$

If $\tan\chi<\frac{\gamma}{r}$, the plane is inclined to the side of the cone, and cuts only one sheet of the surface; in this case, the curve is an ellipse.

If $\tan\chi=\frac{\gamma}{r}$, the plane is parallel to the side of the cone, and the curve is a parabola.

If $\tan\chi>\frac{\gamma}{r}$, the plane cuts both sheets of the surface, and the curve is an hyperbola.

The equation to the intersection of the oblique cone

$$\gamma^2(x^2+y^2)+(a^2-r^2)z^2-2a\gamma xz+2ar^2z+\gamma^2r^2=0,$$

and a plane situated as above is

$$\{(\gamma\cos\chi-a\sin\chi)^2-(r\sin\chi)^2\}\,y_1^2+\gamma^2x_1^2+2ar^2\cos\chi\cdot y_1-\gamma^2r^2=0.$$

The curve is an ellipse, a parabola, or an hyperbola, according as $\tan\chi<, =,$ or $>\frac{\gamma}{r+a}$.

If the section is a circle, $\tan\chi=\dfrac{2a\gamma}{a^2-\gamma^2-r^2}$.

(*H. A. G.* 335—46; *G. G. A.* Ch. xv; *Biot*, Ch. vi.)

SURFACES OF THE SECOND ORDER.

(38.) The general form of the equation to a surface of the second order is

$$Ax^2+By^2+Cz^2+2(A_1yz+B_1xz+C_1xy+A_2x+B_2y+C_2z)+D=0.$$

If the axes are parallel to any system of *conjugate diametral planes*, the equation becomes

$$Ax^2+By^2+Cz^2+2(A_2x+B_2y+C_2z)+D=0. \qquad (1)$$

A straight line cannot intersect a surface of the second order in more than two points.

The equation to a diametral plane, which is the locus of the points of bisection of all chords parallel to the line

$$x = mz, \qquad y = nz, \text{ is}$$

$$(Am + C_1 n + B_1)x + (C_1 m + Bn + A_1)y + (B_1 m + A_1 n + C)z$$
$$+ A_2 m + B_2 n + C_2 = 0.$$

The number of systems of conjugate diametral planes is unlimited; of these, however, only one system can be rectangular.

The co-ordinates of the centre are

$$x = \frac{A_2(A_1^2 - BC) + B_2(CC_1 - A_1B_1) + C_2(BB_1 - A_1C_1)}{ABC + 2A_1B_1C_1 - AA_1^2 - BB_1^2 - CC_1^2}.$$

$$y = \frac{B_2(B_1^2 - AC) + C_2(AA_1 - B_1C_1) + A_2(CC_1 - A_1B_1)}{ABC + 2A_1B_1C_1 - AA_1^2 - BB_1^2 - CC_1^2}.$$

$$z = \frac{C_2(C_1^2 - AB) + A_2(BB_1 - A_1C_1) - B_2(AA_1 - B_1C_1)}{ABC + 2A_1B_1C_1 - AA_1^2 - BB_1^2 - CC_1^2}.$$

If the surface is finite, then

$$A_1^2 - BC < 0, \quad B_1^2 - AC < 0, \quad C_1^2 - AB < 0,$$
$$AA_1^2 + BB_1^2 + CC_1^2 - ABC - 2A_1B_1C_1 < 0.$$

If the surface has not a centre, then

$$AA_1^2 + BB_1^2 + CC_1^2 = ABC + 2A_1B_1C_1.$$

(39.) *Surfaces of revolution.* If the axis of revolution coincides with one of the co-ordinate axes, as x, the general equation to a surface of revolution is

$$y^2 + z^2 = f(x).$$

The conditions which determine a surface of the second order to be a surface of revolution are

$$A_1C_1(C - A) - B_1(C_1^2 - A_1^2) = 0,$$
$$B_1C_1(C - B) - A_1(C_1^2 - B_1^2) = 0:$$

also A_1, B_1, C_1 must all have the same sign.

The equations to the axis of revolution are

$$x = \frac{C_1}{A_1}z, \qquad y = \frac{C_1}{B_1}z.$$

The equation to the *paraboloid* is $x^2+y^2=4mz$;

. . . . *spheroid* . . . $\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2}=1$;

. . . . *hyperboloid* . . $\frac{x^2+y^2}{a^2}-\frac{z^2}{b^2}=-1$.

(*H. A. G.* 347—58; *G. G. A.* 194—9; *Biot*, 312—21.)

(40.) *Surfaces which have a centre.* If the origin is transferred to the centre, the general equation becomes

$$Ax^2+By^2+Cz^2+2(A_1yz+B_1xz+C_1xy)+D=0.$$

If the axes coincide with any system of conjugate diameters, the equation is reduced to

$$Ax^2+By^2+Cz^2+D=0,$$

or
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1;$$

a, b, c being the semi-axes; in which case, the co-ordinates are rectangular.

Let a_1, b_1, c_1 be any system of conjugate diameters, then,

$$a_1^2+b_1^2+c_1^2=a^2+b^2+c^2.$$

If the surface be referred to the diameters a_1, b_1, c_1, then

$$(a_1b_1.\sin x,y)^2+(a_1c_1.\sin x,z)^2+(b_1c_1.\sin y,z)^2=a^2b^2+a^2c^2+b^2c^2.$$

and
$$a^2b^2c^2=a_1^2b_1^2c_1^2\{1-2(\cos x,y.\cos x,z.\cos y,z) \\ -(\cos x,y)^2-(\cos x,z)^2-(\cos y,z)^2\}.$$

(41.) The species of the class of surfaces defined by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

depend on the signs of a^2, b^2, and c^2; the equation to

the ellipsoid is . . . . $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$;

the hyperboloid of one sheet, $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$;

the hyperboloid of two sheets, $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$.

In the ellipsoid, the three traces, or principal sections are ellipses, their equations are

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \text{ the trace on } yz;$$

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1, \text{ . . . . } xz;$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ . . . . } xy.$$

If two of the quantities a^2, b^2, c^2 are equal, the surface is a spheroid, if they are all equal, a sphere.

In the hyperboloid of one sheet, the principal sections are

$$\text{an ellipse, } \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

$$\text{an hyperbola, } \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1,$$

$$\text{. . . . . } \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

The conical surface the equation to which is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0,$$

is an asymptote to the hyperboloid.

In the hyperboloid of two sheets the principal sections are

$$\text{an hyperbola, } \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

$$\text{. . . . . } \quad \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1,$$

$$\text{an imaginary curve, } \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1.$$

The plane yz does not meet the surface.

The equation to the asymptotic cone is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$$

(*H. A G.* 359—81; *G. G. A.* 200—24; *Biot*, 322—30.)

(42.) *Surfaces which have not a centre.* If the origin is at the vertex, and one of the co-ordinate axes, as x, coincides with the axis of the surface, the general form of the equation is

$$My^2 + Nz^2 + Px = 0.$$

This class consists of two species,

the elliptic paraboloid, $ny^2 + mz^2 = 4mnx$,
the hyperbolic paraboloid, $ny^2 - mz^2 = 4mnx$.

The principal sections of the elliptic paraboloid are

a parabola, $y^2 = 4mx$,
. . . . $z^2 = 4nx$,
a point, $ny^2 + mz^2 = 0$.

This surface will be generated by the parabola $z^2 = 4nx$ moving parallel to itself so that its vertex may describe the parabola $y^2 = 4mx$.

The principal sections of the hyperbolic paraboloid are

a parabola, $y^2 = 4mx$,
. . . . . . $z^2 = -4nx$,
two straight lines, $ny^2 - mz^2 = 0$.

All sections parallel to xy and xz are parabolas, and parallel to yz, hyperbolas.

The two planes defined by the equation $ny^2 - mz^2 = 0$ are asymptotes to the surface.

This surface will be generated by the parabola $z^2 = -4nx$ moving parallel to itself, so that its vertex may describe the parabola $y^2 = 4mx$.

(*H. A. G.* 382—95; *G. G. A.* 225—31; *Biot*, 331—4.)

The Intersection of a Surface of the Second Order and a Plane.

(43.) Let the surface be referred to its centre and axes, the equation is $Ax^2 + By^2 + Cz^2 + D = 0$.

The equation to the line of intersection is

$$\{A(\cos\phi)^2 + B(\sin\phi)^2\}\, x^2 + 2(A - B)\sin\phi.\cos\phi.\cos\chi.xy$$
$$+ \{(\cos\chi)^2.\overline{A(\sin\phi)^2 + B(\cos\phi)^2} + C(\sin\chi)^2\}\, y^2 + D = 0;$$

the equation to a curve of the second order, which will be an ellipse, a parabola, or an hyperbola, according as

$$AB(\cos\chi)^2 + C(\sin\chi)^2\{B(\sin\phi)^2 + A(\cos\phi)^2\} >,\ =,\ \text{or} < 0.$$

In the ellipsoid, every section is an ellipse, or one of its varieties.

In the hyperboloids, the section may be either an ellipse, a parabola, or an hyperbola.

(44.) If the surface have not a centre, the equation is

$$My^2 + Nz^2 + Px = 0.$$

The equation to the line of intersection is

$$\{M(\cos\phi.\cos\chi)^2 + N(\sin\chi)^2\}\,y^2 + N(\sin\phi)^2x^2$$
$$-2M\sin\phi.\cos\phi.\cos\chi.xy + P\cos\phi.\cos\chi.y + P\cos\phi.x = 0;$$

the equation to a curve of the second order, which will be an ellipse, a parabola, or an hyperbola according as

$$MN(\sin\phi.\sin\chi)^2 >,\ =,\ \text{or} < 0.$$

In the elliptic paraboloid the section is an ellipse, and in the hyperbolic paraboloid an hyperbola, except when $\phi = 0$, or $\chi = 0$, in which case the section is a parabola.

(45.) If the surface has a centre, the section will be a circle, when

$$A(\sin\phi)^2 + B(\cos\phi)^2 = C(\sin\chi)^2 + (\cos\chi)^2\{A(\sin\phi)^2 + B(\cos\phi)^2\},$$
$$2(A - B)\sin\phi.\cos\phi.\cos\chi = 0.$$

These equations will be fulfilled if the plane is

perpendicular to yz, and $\tan\chi = \pm\left(\dfrac{B-A}{A-C}\right)^{\frac{1}{2}}$,

. . . . xz, ... $= \pm\left(\dfrac{A-B}{B-C}\right)^{\frac{1}{2}}$,

. . . . xy, ... $= \pm\left(\dfrac{A-C}{C-B}\right)^{\frac{1}{2}}$;

of which three quantities only one is possible.

If the quantities a^2, b^2, c^2, be arranged in the order of magnitude, the section intersects the mean axis $2b$ in the ellipsoid, and hyperboloid of two sheets, and the greatest axis $2a$ in the hyperboloid of one sheet.

The inclination of the section to the plane xy is

in the ellipsoid, $\tan^{-1} \pm \dfrac{c}{a}\left(\dfrac{a^2-b^2}{b^2-c^2}\right)^{\frac{1}{2}}$,

. . . . hyperboloid of one sheet, $\tan^{-1} \pm \dfrac{c}{b}\left(\dfrac{a^2-b^2}{a^2+c^2}\right)^{\frac{1}{2}}$,

. . . . two sheets, $\tan^{-1} \pm \dfrac{c}{a}\left(\dfrac{a^2+b^2}{b^2-c^2}\right)^{\frac{1}{2}}$.

(46.) If the surface has not a centre, the section will be a circle, when

$$M(\cos\chi.\cos\phi)^2 + N\{(\sin\chi)^2 - (\sin\phi)^2\} = 0,$$
$$2M\sin\phi.\cos\phi.\cos\chi = 0.$$

These conditions will be fulfilled if the plane is

perpendicular to xz, and $\sin\chi = \pm\left(\dfrac{N}{M}\right)^{\frac{1}{2}}$,

. . . . xy, and $\sin\phi = \pm\left(\dfrac{M}{N}\right)^{\frac{1}{2}}$.

(*H. A. G.* 396—401; *Biot*, 332—4.)

THE TANGENT PLANE.

(47.) If the equation to the surface is

$$Ax^2 + By^2 + Cz^2 + 2(A_2x + B_2y + C_2z) + D = 0;$$

the equation a plane touching the surface at the point (x_1, y_1, z_1) is

$$(Ax_1 + A_2)x + (By_1 + B_2)y + (Cz_1 + C_2)z + A_2x_1 + B_2y_1 + C_2z_1 + D = 0.$$

If $x - x_1 = a(z - z_1)$, $y - y_1 = b(z - z_1)$ are the equations to a line passing through two points (x, y, z), (x_1, y_1, z_1), the point (x_2, y_2, z_2) in which the plane passing through these points touches the surface may be determined from the equations

$$Ax_2^2 + By_2^2 + Cz_2^2 + 2(A_2x_2 + B_2y_2 + C_2z_2) + D = 0,$$
$$(Ax_2 + A_2)x_1 + (By_2 + B_2)y_1 + (Cz_2 + C_2)z_1$$
$$+ A_2x_2 + B_2y_2 + C_2z_2 + D = 0,$$
$$Aax_2 + Bby_2 + Cz_2 = 0.$$

If the equation to the surface is

$$Ax^2 + By^2 + Cz^2 + D = 0,$$

the equation to the tangent plane is

$$Axx_1 + Byy_1 + Czz_1 + D = 0;$$

and the equations to the normal are

$$x - x_1 = \frac{Ax_1}{Cz_1}(z - z_1), \qquad y - y_1 = \frac{By_1}{Cz_1}(z - z_1).$$

If the equation to the surface is

$$My^2 + Nz^2 + Px = 0,$$

the equation to the tangent plane is

$$Myy_1 + Nzz_1 + \tfrac{1}{2}P(x + x_1) = 0.$$

If three planes perpendicular to each other are tangents to the surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

the locus of their intersection is the sphere

$$x^2 + y^2 + z^2 = a^2 + b^2 + c^2.$$

If a conical surface circumscribes a surface of the second order, the line of contact is a plane curve.

If any number of planes passing through a given point intersect a surface of the second order, and at the lines of intersection conical surfaces be circumscribed, the locus of their vertices is a plane.

(*H. A. G.* 364, 85; *G. G. A.* Ch. xx; *Biot*, 335—8.)

DIFFERENTIAL CALCULUS.

(1.) *Differentiation of algebraic functions.*

$d_x(u_1 + u_2 + \&\text{c.} + u_n + \text{const.}) = d_x u_1 + d_x u_2 + \&\text{c. } d_x u_n.$ [α]

$d_x au = a d_x u,$ if a is independent of x.

$d_x x^m = m x^{m-1}.$

$$d_x u_1 u_2 = u_1 d_x u_2 + u_2 d_x u_1 = u_1 u_2 \left(\frac{d_x u_1}{u_1} + \frac{d_x u_2}{u_2}\right).$$

$$d_x u_1 u_2 \ldots u_n = u_1 u_2 \ldots u_n \left(\frac{d_x u_1}{u_1} + \frac{d_x u_2}{u_2} + \&\text{c.} + \frac{d_x u_n}{u_n}\right). \quad {}^{\beta}$$

$$d_x \frac{u}{v} = \frac{v d_x u - u d_x v}{v^2} = \frac{u}{v}\left(\frac{d_x u}{u} - \frac{d_x v}{v}\right).$$

$$d_x \frac{u_1 u_2 \ldots u_m}{v_1 v_2 \ldots v_n} = \frac{u_1 u_2 \ldots u_m}{v_1 v_2 \ldots v_n} \left\{\frac{d_x u_1}{u_1} + \frac{d_x u_2}{u_2} + \&\text{c.} + \frac{d_x u_m}{u_m} - \frac{d_x v_1}{v_1} - \frac{d_x v_2}{v_2} - \&\text{c.} - \frac{d_x v_n}{v_n}\right\}. \quad {}^{\gamma}$$

[α] $d_x u$ expresses the same quantity which has usually been denoted by $\dfrac{du}{dx}$. (*See L. C. D.* Vol. II. p. 527.)

$$d_x\left(\overset{n}{\mathrm{S}}_m u_m + \text{const.}\right) = \overset{n}{\mathrm{S}}_m d_x u_m.$$

[β] $$d_x \overset{n}{\mathrm{P}}_m u_m = \overset{n}{\mathrm{P}}_m u_m \cdot \overset{n}{\mathrm{S}}_m \frac{d_x u_m}{u_m}.$$

[γ] $$d_x \frac{\overset{m}{\mathrm{P}}_r u_r}{\overset{n}{\mathrm{P}}_r v_r} = \frac{\overset{m}{\mathrm{P}}_r u_r}{\overset{n}{\mathrm{P}}_r v_r} \left\{\overset{m}{\mathrm{S}}_r \frac{d_x u_r}{u_r} - \overset{n}{\mathrm{S}}_r \frac{d_x v_r}{v_r}\right\}.$$

(2.) *Differentiation of exponential functions.*

$$d_x \log_\epsilon x = \frac{1}{x}. \qquad d_x \log_a x = \frac{\log_a \epsilon}{x}.$$

$$d_x \epsilon^x = \epsilon^x. \qquad d_x a^x = \log_\epsilon a \,.\, a^x.$$

$$d_x \sin x = \cos x. \qquad d_x \cos x = -\sin x.$$

$$d_x \tan x = (\sec x)^2. \qquad d_x \cot x = -(\mathrm{cosec}\, x)^2.$$

$$d_x \sec x = \tan x \,.\, \sec x. \qquad d_x \mathrm{cosec}\, x = -\cot x \,.\, \mathrm{cosec}\, x.$$

$$d_x \mathrm{vers}\, x = \sin x. \qquad d_x \mathrm{covers}\, x = -\cos x.$$

$$d_x \sin^{-1}\frac{x}{a} = \frac{a}{(a^2 - x^2)^{\frac{1}{2}}}. \qquad d_x \cos^{-1}\frac{x}{a} = \frac{-a}{(a^2 - x^2)^{\frac{1}{2}}}.$$

$$d_x \tan^{-1}\frac{x}{a} = \frac{a^2}{a^2 + x^2}. \qquad d_x \cot^{-1}\frac{x}{a} = \frac{-a^2}{a^2 + x^2}.$$

$$d_x \sec^{-1}\frac{x}{a} = \frac{a^2}{x(x^2 - a^2)^{\frac{1}{2}}}. \qquad d_x \mathrm{cosec}^{-1}\frac{x}{a} = \frac{-a^2}{x(x^2 - a^2)^{\frac{1}{2}}}.$$

$$d_x \mathrm{vers}^{-1}\frac{x}{a} = \frac{a}{(2ax - x^2)^{\frac{1}{2}}}. \qquad d_x \mathrm{covers}^{-1}\frac{x}{a} = \frac{-a}{(2ax - x^2)^{\frac{1}{2}}}.$$

SUCCESSIVE DIFFERENTIATION.

(3.) $d_x x^n = nx^{n-1}.$

$d_x^2 x^n = n(n-1)x^{n-2}.$

&c. = &c.

$d_x^m x^n = n(n-1)\ldots(n-m+1)x^{n-m}.$

$d_x^n x^n = n(n-1)\ldots 2 \,.\, 3 \,.\, 1.$

$d_x uv = ud_x v + vd_x u.$

$d_x^2 uv = ud_x^2 v + 2d_x u \,.\, d_x v + vd_x^2 u.$

$d_x^3 uv = ud_x^3 v + 3d_x u \,.\, d_x^2 v + 3d_x^2 u \,.\, d_x v + vd_x^3 u.$

&c. = &c.

$$d_x^n uv = d_x^n u \,.\, v + nd_x^{n-1} u \,.\, d_x v + \frac{n(n-1)}{1 \,.\, 2} d_x^{n-2} u \,.\, d_x^2 v + \text{\&c.}$$
$$+ \frac{n(n-1)\ldots(n-r+1)}{1 \,.\, 2 \ldots r} d_x^{n-r} u \,.\, d_x^r v + \text{\&c.} + ud_x^n v.$$ [α]

[α] $d_x^n{:}uv = \overset{n+1}{\mathrm{S}}_m d_x^{n-m+1}{:}u \,.\, d_x^{m-1}{:}v.$

(4.) $d_x^n a^x = (\log_\varepsilon a)^n a^x.$

$d_x^n \log_\varepsilon x = (-1)^{n-1} 1 . 2 . 3 \ldots (n-1) x^{-n}.$

$$\begin{array}{llll} d_x \sin x = \cos x, & & d_x^2 \sin x = -\sin x, & \\ d_x^3 \sin x = -\cos x, & & d_x^4 \sin x = \sin x, & \\ \&c. = \&c. & & \&c. = \&c. & \\ d_x^{2n-1} \sin x = (-1)^{n-1} \cos x, & & d_x^{2n-2} \sin x = (-1)^{n-1} \sin x. & \end{array}$$

$d_x \tan x = 1 + (\tan x)^2,$

$d_x^2 \tan x = 2 \tan x \{1 + (\tan x)^2\},$

$d_x^3 \tan x = 2 \{1 + (\tan x)^2\} \{1 + 3 (\tan x)^2\}.$

&c. = &c. [β]

$d_x \sec x = \sec x . \overline{(\sec x)^2 - 1}\big]^{\frac{1}{2}}.$

$d_x^2 \sec x = \sec x . \{2 (\sec x)^2 - 1\}.$

$d_x^3 \sec x = \sec x . \{2 . 3 (\sec x)^2 - 1\} . \overline{(\sec x^2) - 1}\big]^{\frac{1}{2}}.$

&c. = &c.

These differential coefficients may be adapted to cos x, cot x, and cosec x by substituting these quantities for sin x, tan x, and sec x respectively, and changing the signs of those coefficients the index of which is an odd number.

(5.) $d_x \sin^{-1} x = (1 - x^2)^{-\frac{1}{2}}.$

$d_x^2 \sin^{-1} x = x (1 - x^2)^{-\frac{3}{2}}.$

$d_x^3 \sin^{-1} x = (1 - x^2)^{-\frac{3}{2}} + 3 x^2 (1 - x)^{2 - \frac{5}{2}}.$

&c. = &c.

[β] d_x^{2m-1}:tan x

$$= \overset{m}{\mathrm{S}}_n \frac{(-1)^{m-n} \cos x}{\lfloor\!\underline{2m - 2n + 1}} \varpi^{2n-2} . a^{-1} + \overset{m}{\mathrm{S}}_n \frac{(-1)^{m-n} \sin x}{\lfloor\!\underline{2m - 2n}} \varpi^{2n-1} . a^{-1};$$

d_x^{2m-2}:tan x

$$= \overset{m}{\mathrm{S}}_n \frac{(-1)^{m-n} \sin x}{\lfloor\!\underline{2m - 2n}} \varpi^{2n-2} . a^{-1} + \overset{m-1}{\mathrm{S}}_n \frac{(-1)^{m-n-1} \cos x}{\lfloor\!\underline{2m - 2n - 1}} \varpi^{2n-1} . a^{-1}.$$

$$\left\{ a_{2r-2} = \frac{(-1)^{r-1} \cos x}{\lfloor\!\underline{2r - 2}}, \ a_{2r-1} = \frac{(-1)^r \sin x}{\lfloor\!\underline{2r - 1}} \right\}.$$

$d_x \tan^{-1}x = (1+x^2)^{-1}$.

$d_x^2 \tan^{-1}x = -2x(1+x^2)^{-2}$.

$d_x^3 \tan^{-1}x = -2(1+x^2)^{-2} + 2^3.x^2(1+x^2)^{-3}$.

&c. = &c.

$d_x \sec^{-1}x = x^{-1}(x^2-1)^{-\frac{1}{2}}$.

$d_x^2 \sec^{-1}x = -x^{-2}(x^2-1)^{-\frac{1}{2}} - (x^2-1)^{-\frac{3}{2}}$.

$d_x^3 \sec^{-1}x = 2x^{-3}(x^2-1)^{-\frac{1}{2}} + x^{-1}(x^2-1)^{-\frac{3}{2}} + 3x(x^2-1)^{-\frac{5}{2}}$.

The coefficients of $\cos^{-1}x$, $\cot^{-1}x$, $\operatorname{cosec}^{-1}x$ may be found from these as above. (*L. D. C.* 34—48.)

(6.) $d_x\phi(u) = d_u\phi(u).d_xu$. [α] (*L. C. D.* 11; *L. D. C.* 16.)

DIFFERENTIATION OF FUNCTIONS OF SUPERIOR ORDERS.

(7.) Let $d_x\phi(x) = \phi'(x)$, then

$d_x\phi^n(x) = \phi'(x).\phi'\phi(x).\phi'\phi^2(x)\ldots\phi'\phi^{n-1}(x)$. [β]

$d_x\sin^n x = \cos x.\cos\sin x.\cos\sin^2 x\ldots\cos\sin^{n-1}x$. [γ]

[α] $d_x^n{:}\phi(u) = \overset{n}{\mathrm{S}}_m d_u^m\phi(u)\dfrac{\varpi^{n-m}.a^m}{m}$, $\{a_{r-1} = d_x^r{:}u\}$. *See Appendix.*

A few applications of this important theorem may here be added:

$d_x^n{:}u^m = \varpi^n.a^m$, $\{a_{r-1} = d_x^{r-1}{:}u\}$.

$d_x^n{:}\epsilon^u = \epsilon^u.\overset{n}{\mathrm{S}}_m \dfrac{\varpi^{n-m}.a^m}{\lfloor\underline{m}}$, $\{a_{r-1} = d_x^r{:}u\}$.

$d_x^n{:}\sin u$

$$= \overset{r}{\mathrm{S}}_m(-1)^{m-1}\cos u\frac{\varpi^{n-2m+1}.a^{2m-1}}{\lfloor\underline{2m-1}} + \overset{n-r}{\mathrm{S}}_m(-1)^m\sin u\frac{\varpi^{n-2m}.a^{2m}}{\lfloor\underline{2m}};$$

$d_x^n{:}\cos u$

$$= \overset{r}{\mathrm{S}}_m(-1)^m\sin u\frac{\varpi^{n-2m+1}.a^{2m-1}}{\lfloor\underline{2m-1}} + \overset{n-r}{\mathrm{S}}_m(-1)^m\cos u\frac{\varpi^{n-2m}.a^{2m}}{\lfloor\underline{2m}}.$$

$\{a_{m-1} = d_x^m{:}u$; $r = \frac{1}{2}n$, if n is even, $r = \frac{1}{2}(n-1)$, if n is odd.$\}$

[β] $d_x\phi^n(x) = \overset{n}{\mathrm{P}}_m\phi'\phi^{m-1}(x)$.

[γ] $d_x\sin^n x = \overset{n}{\mathrm{P}}_m\cos\sin^{m-1}x$.

$d_x \cos^n x = (-1)^n \sin x . \sin\cos x . \sin\cos^2 x \ldots \sin\cos^{n-1} x.$ [δ]

$d_x \log_\epsilon^n x = x^{-1} . (\log_\epsilon x)^{-1} . (\log_\epsilon^2 x)^{-1} \ldots (\log_\epsilon^{n-1} x)^{-1}.$ [ε]

DEVELOPMENT OF FUNCTIONS.

(8.) $d_x f(x+y) = d_y f(x+y)$.

Taylor's Theorem. Let $E_x(u)$ represent what u becomes when $x + Dx$ is substituted for x, then

$$E_x(u) = u + d_x u \frac{Dx}{1} + d_x^2 u \frac{(Dx)^2}{1.2} + d_x^3 u \frac{(Dx)^3}{1.2.3} + \&c. \quad [\theta]$$

If the n first terms of this series be taken, the limits of error are the greatest and least values of $d_x^n u \dfrac{(Dx)^n}{1.2.3 \ldots n}$ provided there be no value of x between x, and $x + Dx$ that renders u or any of its differential coefficients infinite.

(*L. C. D.* 169.)

Maclaurin's Theorem. Let $(u)_{x=0}$, $d_{x=0}^n u$ represent the values of u, $d_x^n u$, respectively, when $x = 0$, then

$$u = (u)_{x=0} + d_{x=0} u . \frac{x}{1} + d_{x=0}^2 u . \frac{x^2}{1.2} + d_{x=0}^3 u . \frac{x^3}{1.2.3} + \&c. \quad [\mu]$$

[δ] $d_x \cos^n x = (-1)^n \overset{n}{\mathrm{P}}_m \sin\cos^{m-1} x.$

[ε] $d_x \log_\epsilon^n x = \overset{n}{\mathrm{P}}_m (\log_\epsilon^{m-1} x)^{-1}.$

[θ] $E_x u = \overset{\infty}{\mathrm{S}}_m (Dx)^{m-1} . d_x^{m-1} : u.$

Let $E_x u - u = D_x u$, then

$$Dx = \frac{D_x u}{d_x u} + \overset{\infty}{\mathrm{S}}_n \overline{D_x u}|^{n+1} . \overset{n}{\mathrm{S}}_m \frac{(-1)^{n-m+1}}{\underline{|n-m+1}} . \underline{|2n-m+1}_{n-m} . \frac{\varpi^{m-1} . a^{n-m+1}}{(d_x u)^{2n-m+2}},$$

where $a_{r-1} = d_x^{r+1} : u.$

[μ] $u = \overset{\infty}{\mathrm{S}}_m x^{m-1} . d_{x=0}^{m-1} : u.$

(9.) Taylor's theorem applied to two variables.

$$\begin{aligned}E_{x,y}(u) = u &+ \{d_x u.Dx + d_y u.Dy\}\\ &+ \frac{1}{1.2}\{d_x^2 u(Dx)^2 + 2d_x d_y u.Dx.Dy + d_y^2 u(Dy)^2\}\\ &+ \frac{1}{1.2.3}\{d_x^3 u(Dx)^3 + 3d_x^2 d_y u.(Dx)^2 Dy\\ &\qquad + 3d_x d_y^2 u Dx.(Dy)^2 + d_y^3 u(Dy)^3\} + \&c. \quad [\alpha]\end{aligned}$$

$d_x^m d_y^n u = d_y^n d_x^m u.$ (*L. C. D.* 25—30.)

(10). *Laplace's Theorem.* Let $y = \psi\{z + x.\phi(y)\}$, in which z is independent of x and y, then

$$\begin{aligned}f(y) = f\psi(z) &+ \frac{x}{1}\{\phi\psi(z).d_z f\psi(z)\} + \frac{x^2}{1.2}.d_z\{\left.\overline{\phi\psi(z)}\right|^2.d_z f\psi(z)\}\\ &+ \frac{x^3}{1.2.3}.d_z^2\{\left.\overline{\phi\psi(z)}\right|^3.d_z f\psi(z)\} + \&c. \quad [\beta]\end{aligned}$$

Lagrange's Theorem. Let $y = z + x.\phi(y)$, then

$$\begin{aligned}f(y) = f(z) &+ \frac{x}{1}\{\phi(z).d_z f(z)\} + \frac{x^2}{1.2}.d_z\{\left.\overline{\phi(z)}\right|^2.d_z f(z)\}\\ &+ \frac{x^3}{1.2.3}.d_z^2\{\left.\overline{\phi(z)}\right|^3.d_z f(z)\} + \&c. \quad [\gamma]\end{aligned}$$

This theorem is a particular case of the former, in which

$$\psi\{z + x.\phi(y)\} = z + x.\phi(y).$$

[α] $E_{x,y}(u) = \overset{\infty}{\mathrm{S}}_m \overset{m}{\mathrm{S}}_n (Dx)^{m-n}.(Dy)^{n-1}.d_x^{\,m-n}{:}d_y^{\,n-1}{:}u.$

If a third variable z be supposed to receive an increment, then

$$E_{x,y,z}(u) = \overset{\infty}{\mathrm{S}}_m \overset{m}{\mathrm{S}}_n \overset{n}{\mathrm{S}}_r (Dx)^{m-n}.(Dy)^{n-r}.(Dz)^{r-1}.d_x^{\,m-n}{:}d_y^{\,n-r}{:}d_z^{\,r-1}{:}u.$$

Maclaurin's theorem applied to a function of two variables.

$$\begin{aligned}u &= \overset{\infty}{\mathrm{S}}_m x^{m-1}.\overset{\infty}{\mathrm{S}}_n y^{n-1}.d_{x=0}^{m-1}{:}d_{x=0}^{n-1}{:}u,\\ &= \overset{\infty}{\mathrm{S}}_m \overset{m}{\mathrm{S}}_n x^{m-n}.y^{n-1}.d_{x=0}^{m-n}{:}d_{x=0}^{n-1}{:}u.\end{aligned}$$

(*Jarrett, Camb. Phil. Trans.* Vol. III, Pt. 1.)

[β] $f(y) = f\psi(z) + \overset{\infty}{\mathrm{S}}_m \frac{x^m}{\lfloor\underline{m}}.d_z^{m-1}\{\left.\overline{\phi\psi(z)}\right|^m.d_z f\psi(z)\}.$

[γ] $f(y) = f(z) + \overset{\infty}{\mathrm{S}}_m \frac{x^m}{\lfloor\underline{m}}.d_z^{m-1}\{\left.\overline{\phi(z)}\right|^m.d_z f(z)\}.$

If $f(y)=y$, then

$$y=z+\frac{x}{1}\cdot\phi(z)+\frac{x^2}{1.2}\cdot d_z\overline{\phi(z)|}^2+\frac{x^3}{1.2.3}\cdot d_z^2\overline{\phi(z)|}^3+\&c.\ ^{\delta}$$

(*L. C. D.* 107—9; *Tr. L.* Note E; *L. D. C.* 49—61.)

IMPLICIT FUNCTIONS.

(11.) Let $d_x u=M$, $d_y u=N$, then

$$M+Nd_x y=0,$$

from which equation the value of $d_x y$ may be found. By successive differentiation, the following equations may be obtained, in which u_1, u_2, &c. are functions of x and y:

$$u_1+u_2 d_x y+u_3(d_x y)^2+Nd_x^2 y=0,$$

$$u_4+u_5 d_x y+u_6(d_x y)^2+u_7(d_x y)^3+(u_8+u_9 d_x y)\,d_x^2 y+Nd_x^3 y=0,$$

$$\&c. \qquad\qquad \&c.$$

from which the values of $d_x^2 y$, $d_x^3 y$, &c. may be obtained.

(*L. C. D.* 41—8; *L. D. C.* 100—3.)

TRANSFORMATION OF THE INDEPENDENT VARIABLE.

(12.) $d_y x=\dfrac{1}{d_x y}$, $\qquad d_y^2 x=\dfrac{d_x^2 y}{(d_x y)^3}$,

$d_y^3 x=\dfrac{3(d_x^2 y)^2-d_x y\,.\,d_x^3 y}{(d_x y)^5}$, $\qquad$ &c. = &c.

If x and y are considered as functions of a third variable t, then

$$d_x y=\frac{d_t y}{d_t x},$$

$$d_x^2 y=\frac{d_t x\,.\,d_t^2 y-d_t^2 x\,.\,d_t y}{(d_t x)^3},$$

$$d_x^3 y=\frac{(d_t x)^2\,.\,d_t^3 y-3\,d_t x\,.\,d_t^2 x\,.\,d_t^2 y+3\,(d_t^2 x)^2\,.\,d_t y-d_t x\,.\,d_t^3 x\,.\,d_t y}{(d_t x)^5},$$

&c. = &c. (*L. C. D.* 57—69.)

[δ] $y=z+\overset{\infty}{\mathrm{S}}_m\dfrac{x^m}{\underline{|m}}\,.\,d_z^{m-1}\overline{\phi(z)|}^m.$

(13.) *Elimination of an arbitrary function.*

Let $z=\phi f(x,y)=\phi(t)$ suppose, then

$$d_x z=d_t\phi(t).d_x t, \qquad \text{and } d_y z=d_t\phi(t).d_y t, \text{ and hence}$$

$$\frac{d_x z}{d_y z}=\frac{d_x f(x,y)}{d_y f(x,y)}.$$

In the same manner, by repeating the differentiation of any given equation, two or more arbitrary functions may be eliminated. (*L. C. D.* 77—9.)

PARTICULAR VALUES OF THE VARIABLE.

(14.) Let $u=0$, when $x=a$, then u must be of the form

$$v(x-a)^m,$$

v being a function of x not containing the factor $x-a$.

$d_x u=v_1(x-a)^{m-1}$.

&c. = &c.

$d_x^n u=v_n(x-a)^{m-n}$,

n being the greatest integer $\ngtr m$.

If $m=n$, then $d_x^n u=v_n$, which is the first differential coefficient that does not vanish, when $x=a$.

If m is not an integer, then $d_x^{n+1}u$ and all succeeding coefficients become infinite, when $x=a$.

Let $u=\frac{0}{0}$ when $x=a$, then

$$u=\frac{v_1(x-a)^m}{v_2(x-a)^p}.$$

Let n, r be the greatest integers $\ngtr m$, p respectively, then when $x=a$, if $n>r$, $u=0$;

. . . . $n<r$, u, and all its differential coefficients are infinite;

. . . . $n=r=m$, $u=\frac{v_1}{0}=\infty$;

. . . . $n=r=p$, $u=\frac{0}{v_2}=0$;

when $x = a$, if $n = r$, and $m > p$, $u = 0$;

. . . . $m = p$, $u = \frac{v_1}{v_2}$;

. . . . $m < p$, $u = \infty$. (*L. C. D.* 143—53.)

If $u = \frac{f(x)}{\phi(x)} = \frac{0}{0}$, when $x = a$, the value of u may be thus found: substitute $a + Dx$ for x in $f(x)$ and $\phi(x)$, and expand in series of ascending powers of Dx, then

$$u = \frac{a_1 (Dx)^{m_1} + a_2 (Dx)^{m_2} + \&c.}{b_1 (Dx)^{n_1} + b_2 (Dx)^{n_2} + \&c.}$$

If $m_1 > n_1$, $u = \frac{a_1 (Dx)^{m_1 - n_1} + a_2 (Dx)^{m_2 - n_1} + \&c.}{b_1 + b_2 (Dx)^{n_2 - n_1} + \&c.}$

$= 0$, when $Dx = 0$.

If $m_1 < n_1$, $u = \frac{a_1 + a_2 (Dx)^{m_2 - m_1} + \&c.}{b_1 (Dx)^{n_1 - m_1} + b_2 (Dx)^{n_2 - m_1} + \&c.}$

$= \infty$, when $Dx = 0$.

If $m_1 = n_1$, $u = \frac{a_1 + a_2 (Dx)^{m_2 - m_1} + \&c.}{b_1 + b_2 (Dx)^{n_2 - n_1} + \&c.}$

$= \frac{a_1}{b_1}$, when $Dx = 0$.

If $u = f(x) . f_1(x) = 0 . \infty$, when $x = a$, put $\phi(x) = \frac{1}{f_1(x)}$,

then $$u = \frac{f(x)}{\phi(x)} = \frac{0}{0}, \text{ when } x = a.$$

If $u = \frac{f(x)}{\phi(x)} = \frac{\infty}{\infty}$, when $x = a$, then

$$u = \frac{\{\phi(x)\}^{-1}}{\{f(x)\}^{-1}} = \frac{0}{0}, \text{ as before:}$$

under which form the value of u may be determined as above.

(*L. C. D.* 12; *L. D. C.* 107.)

(15.) Let $f(x,y)=0$, be an equation cleared of irrational quantities in which y is an implicit function of x, then (Art. 11.)

$$M+Nd_xy=0, \qquad (1)$$

$$u_1+u_2d_xy+u_3(d_xy)^2+Nd_x^2y=0, \qquad (2)$$

and by eliminating d_xy we obtain an equation of the form

$$M_2+Nd_x^2y=0,$$

and generally an equation may be obtained of the form

$$M_n+Nd_x^ny=0.$$

If the values of x and y which satisfy the equations

$$f(x,y)=0, \qquad M_n=0,$$

do not satisfy $N=0$, then $d_x^ny=0$.

If the values of x and y which satisfy the equations

$$f(x,y)=0, \qquad N=0,$$

do not satisfy $M_n=0$, then $d_x^ny=\infty$.

If $f(x,y)=0$, $M=0$, $N=0$ be satisfied by the same values of x and y, then $d_xy=\frac{0}{0}$, the value of which may be obtained from (2) since $N=0$, unless u_1, u_2, u_3 vanish for the same values of x and y, in which case we must differentiate again, and obtain an equation of the form

$$v_1+v_2d_xy+v_3(d_xy)^2+v_4(d_xy)^3=0;$$

and so on until we arrive at an equation the coefficients of which do not vanish for the proposed values of x and y.

If $d_x^ny=\frac{0}{0}$, its values may be similarly found.

If for any system of values of the variables, any differential coefficient has more than one value, then that coefficient obtained from the differential equation of the corresponding order must appear under the form $\frac{0}{0}$.

(*L. C. D.* 135—7; *L. D. C.* 109—13.)

(16.) Let the values of u corresponding to $a+Dx$, $a-Dx$ be u_1, u_{-1} respectively; then if $u-u_1$ and $u-u_{-1}$ have the same sign for all values of x from a to $a \pm Dx$, the value of u when $x=a$ is a *maximum* or *minimum*, according as $u-u_1$ and $u-u_{-1}$ are both positive or both negative.

When $d_x u = 0$, u is a maximum or minimum, according as $d_x^2 u$ is negative or positive.

If the same value of x makes $d_x^2 u = 0$, there is no maximum or minimum unless $d_x^3 u$ also vanishes: and generally there is no maximum or minimum, unless the first differential coefficient that does vanish with any given value of x is of an even order.

Let the value of x which makes $d_x^m u = 0$, make $d_x^{m+1} u = \infty$: $u-u_1$ and $u-u_{-1}$ may be thus developed in ascending powers of Dx;

$$u-u_{-1} = a_1(-Dx)^{m_1} + a_2(-Dx)^{m_2} + \&c.$$
$$u-u_1 = a_1(Dx)^{m_1} + a_2(Dx)^{m_2} + \&c.$$

then if any of the indices m_1, m_2, &c. have an *even denominator*, or if m_1 has an *odd numerator*, there is no maximum or minimum.

If m_1 has an even numerator, u will be a maximum or minimum, according as a_1 is positive or negative.

(*L. C. D.* 154—64.)

(17.) Let $u=f(x,y)$: the values of x and y which make $d_x u = 0$, $d_y u = 0$, give a maximum or minimum value of u, provided $d_x^2 u$, $d_x d_y u$, and $d_y^2 u$ do not vanish for the same values of x and y, and that

$$d_x^2 u \,.\, d_y^2 u > (d_x d_y u)^2.$$

$d_x^2 u$, and $d_y^2 u$ must evidently have the same sign: and generally there will be no maximum or minimum, unless the first series of differential coefficients which do not vanish for any given values of x and y be of an even order; and unless the series of vanishing differential coefficients might be the coefficients of an equation of the same dimensions, which has no possible root.

(*Garnier*, *Diff. Calc.* Ch. xxv.)

INTEGRAL CALCULUS.

FUNDAMENTAL FORMULÆ.

(1.) $\int_x d_x u = u + const.$

$\int_x au = a\int_x u$, if a is independent of x.

$\int_x (u_1 + u_2 + \&c. + u_n) = \int_x u_1 + \int_x u_2 + \&c. + \int_x u_n.$

$\int_x u . d_x v = uv - \int_x v . d_x u.$

$$\int_x \frac{d_x u}{v} = \frac{u}{v} + \int_x \frac{u d_x v}{v^2}.$$

$$\int_x x^m = \frac{x^{m+1}}{m+1} + const.$$

(2.) *Inverse circular functions.*

$$\int_x \frac{a}{(a^2 - x^2)^{\frac{1}{2}}} = \sin^{-1}\frac{x}{a} + const.$$

$$\int_x \frac{-a}{(a^2 - x^2)^{\frac{1}{2}}} = \cos^{-1}\frac{x}{a} + const.$$

$$\int_x \frac{a^2}{a^2 + x^2} = \tan^{-1}\frac{x}{a} + const.$$

$$\int_x \frac{-a^2}{a^2 + x^2} = \cot^{-1}\frac{x}{a} + const.$$

$$\int_x \frac{a^2}{x(x^2 - a^2)^{\frac{1}{2}}} = \sec^{-1}\frac{x}{a} + const.$$

$$\int_x \frac{-a^2}{x(x^2 - a^2)^{\frac{1}{2}}} = \operatorname{cosec}^{-1}\frac{x}{a} + const.$$

$$\int_x \frac{a}{(2ax - x^2)^{\frac{1}{2}}} = \operatorname{vers}^{-1}\frac{x}{a} + const.$$

$$\int_x \frac{-a}{(2ax - x^2)^{\frac{1}{2}}} = \operatorname{covers}^{-1}\frac{x}{a} + const.$$

(3.) *Logarithmic integrals.*

$$\int_x \frac{1}{x} = \log_e x + const.$$

$$\int_x \frac{1}{(x^2 \pm a^2)^{\frac{1}{2}}} = \log_e \{x + (x^2 \pm a^2)^{\frac{1}{2}}\} + const.$$

$$\int_x \frac{1}{(x^2 \pm 2ax)^{\frac{1}{2}}} = \log_e \{x \pm a + (x^2 \pm 2ax)^{\frac{1}{2}}\} + const.$$

$$\int_x \frac{1}{a^2 - x^2} = \frac{1}{2a} \log_e \left(\frac{a + x}{a - x}\right) + const.$$

$$\int_x \frac{1}{x^2 - a^2} = \frac{1}{2a} \log_e \left(\frac{x - a}{x + a}\right) + const.$$

$$\int_x \frac{1}{x(a^2 \pm x^2)^{\frac{1}{2}}} = \frac{1}{2a} \log_e \left(\frac{a \sim (a^2 \pm x^2)^{\frac{1}{2}}}{a + (a^2 \pm x^2)^{\frac{1}{2}}}\right) + const.$$

DECOMPOSITION OF RATIONAL FRACTIONS.

(4.) Every rational fraction may be reduced to the form

$$\frac{A_m x^m + A_{m-1} x^{m-1} + \&c. + A_1 x + A}{B_n x^n + B_{n-1} x^{n-1} + \&c. + B_1 x + B}, \qquad \left(\frac{U}{V}\right)$$

in which m may be considered less than n.

Let a_1, a_2, &c. a_n be the roots of the equation $V=0$; and [1] let these quantities be possible, and unequal.

If $(U)_{x=a}$, $d_{x=a} V$ respectively represent the values of U, $d_x V$, when $x=a$, then

$$\frac{U}{V} = \frac{(U)_{x=a_1}}{d_{x=a_1} V} \cdot \frac{1}{x - a_1} + \frac{(U)_{x=a_2}}{d_{x=a_2} V} \cdot \frac{1}{x - a_2} + \&c. + \frac{(U)_{x=a_n}}{d_{x=a_n} V} \cdot \frac{1}{x - a_n}.$$

Or thus: let $V = (x - a_1) Q$, and assume

$$\frac{U}{V} = \frac{K_1}{x - a_1} + \frac{P}{Q},$$

then $$K_1 = \left(\frac{U}{Q}\right)_{x=a_1}, \text{ and } P = \frac{U - K_1 Q}{x - a_1}.$$

The other partial fractions may be similarly found: $U - K_r Q$ being always divisible by $x - a_r$.

[2] Let the equation $V=0$ have r equal possible roots.

Assume $\dfrac{U}{V}=\dfrac{K_1}{(x-a)^r}+\dfrac{K_2}{(x-a)^{r-1}}+\&\text{c.}+\dfrac{K_r}{x-a}+\dfrac{P}{Q}$.

$$\text{then } K_1=\left(\frac{U}{Q}\right)_{x=a}, \quad K_2=d_{x=a}\frac{U}{Q}, \quad K_3=\frac{1}{1.2}d^2_{x=a}\frac{U}{Q},$$

$$\&\text{c.}=\&\text{c.} \qquad K_r=\frac{1}{1.2\ldots(r-1)}.d^{r-1}_{x=a}\frac{U}{Q}.$$

Or thus: assume $\dfrac{U-K_1Q}{x-a}=U_1$, then $K_2=\left(\dfrac{U_1}{Q}\right)_{x=a}$

$$\ldots\ldots \frac{U_1-K_2Q}{x-a}=U_2,\ \ldots\ K_3=\left(\frac{U_2}{Q}\right)_{x=a}$$

$$\ldots\ldots \quad \&\text{c.} \quad =\&\text{c.}\ \ldots\ K_r=\left(\frac{U_{r-1}}{Q}\right)_{x=a}.$$

[3] Let the equation $V=0$ have unequal impossible roots, and let

$V=(x^2+c_1x+e_1)Q=(x-a_1+\beta_1\sqrt{-1})(x-a_1-\beta_1\sqrt{-1})Q$,

Q not containing the factor $x^2+c_1x+e_1$.

Assume $$\frac{U}{V}=\frac{K_1+L_1(x+\frac{1}{2}c_1)}{x^2+c_1x+e_1}+\frac{P}{Q},$$

and let $M+N\sqrt{-1}$ be the value of $u-\{K_1+L_1(x+\frac{1}{2}c_1)\}Q$ when $a_1+\beta_1\sqrt{-1}$ is substituted for x; from the equations

$$M=0, \quad N=0,$$

K_1 and L_1 may be obtained, and $P=\dfrac{U-\{K_1+L_1(x+\frac{1}{2}c_1)\}Q}{x^2+c_1x+e_1}$: the other partial fractions, which will be of the same form, may be similarly obtained.

The values of K_1 and L_1 may sometimes be more conveniently obtained by substituting $\dfrac{1}{2\beta_1\sqrt{-1}}.d_xV$ for Q in the quantity $U-\{K_1+L_1(x+\frac{1}{2}c_1)\}Q$, and proceeding as above.

[4] Let the equation $V=0$ have r equal impossible roots.

$$\text{Assume } \frac{U}{V}=\frac{K_1+L_1(x+\frac{1}{2}c)}{(x^2+cx+e)^r}+\frac{K_2+L_2(x+\frac{1}{2}c)}{(x^2+cx+e)^{r-1}}+\text{\&c.}$$

$$+\frac{K_r+L_r(x+\frac{1}{2}c)}{x^2+cx+e}+\frac{P}{Q};$$

$$\text{then } U_1=\frac{U-\{K_1+L_1(x+\frac{1}{2}c)\}Q}{x^2+cx+e},$$

$$U_2=\frac{U_1-\{K_2+L_2(x+\frac{1}{2}c)\}Q}{x^2+cx+e},$$

$$\text{\&c.} = \text{\&c.}$$

$$U_r=\frac{U_{r-1}-\{K_r+L_r(x+\frac{1}{2}c)\}Q}{x^2+cx+e};$$

the constants K_1, L_1, K_2, L_2, &c. may be obtained from the quantities U, U_1, &c. as in the preceding case.

The resolution may in all these cases be effected by the method of indeterminate coefficients, by clearing the equation of fractions, and equating like powers of x; this method is however usually much more laborious.

(*L. C. D.* 380—2; *Hirsch, Int. Tab.*)

(5.) *Decomposition of the fraction* $\dfrac{x^r}{x^n \pm 1}$.

$$\frac{x^r}{x^n+1}=\frac{2}{n}\left\{\frac{\cos\frac{r\pi}{n}-\cos\frac{(r+1)\pi}{n}x}{x^2-2\cos\frac{\pi}{n}x+1}+\frac{\cos 3\frac{r\pi}{n}-\cos 3\frac{(r+1)\pi}{n}x}{x^2-2\cos 3\frac{\pi}{n}x+1}+\dots\right.$$

$$\left.\dots+\frac{\cos(2m-1)r\frac{\pi}{n}-\cos(2m-1)(r+1)\frac{\pi}{n}.x}{x^2-2\cos(2m-1)\frac{\pi}{n}x+1}+\text{\&c.}\right\}$$

to $\frac{1}{2}n$ terms, when n is even.

$$\frac{x^r}{x^n+1}=\frac{1}{n}\cdot\frac{(-1)^r}{x+1}+\frac{2}{n}\left\{\frac{\cos r\frac{\pi}{n}-\cos(r+1)\frac{\pi}{n}x}{x^2-2\cos\frac{\pi}{n}x+1}+\dots\right.$$

$$\left.\dots+\frac{\cos(2m-1)r\frac{\pi}{n}-\cos(2m-1)(r+1)\frac{\pi}{n}x}{x^2-2\cos(2m-1)\frac{\pi}{n}x+1}+\&c.\right\}$$

to $\frac{1}{2}(n-1)$ terms, when n is odd.

$$\frac{x^r}{x^n-1}=\frac{1}{n}\left(\frac{1}{x-1}-\frac{(-1)^r}{x+1}\right)+\frac{2}{n}\left\{\frac{\cos 2(r+1)\frac{\pi}{n}x-\cos 2r\frac{\pi}{n}}{x^2-2\cos 2\frac{\pi}{n}x+1}+\dots\right.$$

$$\left.\dots+\frac{\cos 2m(r+1)\frac{\pi}{n}x-\cos 2mr\frac{\pi}{n}}{x^2-2\cos 2m\frac{\pi}{n}x+1}+\&c.\right\}$$

to $\frac{1}{2}n-1$ terms, when n is even.

$$\frac{x^r}{x^n-1}=\frac{1}{n}\cdot\frac{1}{x-1}+\frac{2}{n}\left\{\frac{\cos 2(r+1)\frac{\pi}{n}x-\cos 2r\frac{\pi}{n}}{1-2\cos 2\frac{\pi}{n}x+1}+\dots\right.$$

$$\left.\dots+\frac{\cos 2m(r+1)\frac{\pi}{n}x-\cos 2mr\frac{\pi}{n}}{x^2-2\cos 2m\frac{\pi}{n}x+1}+\&c.\right\}$$

to $\frac{1}{2}(n-1)$ terms, when n is odd.

The partial fractions of which the fraction $\frac{1}{x^n\pm 1}$ is composed may be obtained from the preceding by putting $r=0$.

Let R denote a rational function.

Differential coefficients of the form

[1] $R\{x, x^{\frac{m}{n}}, x^{\frac{p}{q}}, x^{\frac{r}{s}}, \&c.\}$

may be rendered rational by assuming

$$x = y^{nqs\ldots}$$

[2] $R\{x, (a+bx)^{\frac{m}{n}}, (a+bx)^{\frac{p}{q}}, \&c.\}$;

Assume $a + bx = y^{nq\ldots}$

[3] $R\left\{x, \left(\frac{a+bx}{a_1+b_1x}\right)^{\frac{m}{n}}, \left(\frac{a+bx}{a_1+b_1x}\right)^{\frac{p}{q}}, \&c.\right\}$;

Assume $\dfrac{a+bx}{a_1+b_1x} = y^{nq\ldots}$

[4] $R\{x, (a+bx+cx^2)^{\frac{1}{2}}\}$;

Assume $(a+bx+cx^2) = c(x+y)^2$.

[5] $R\{x, (a+bx-cx^2)^{\frac{1}{2}}\}$;

let $(a+bx-cx^2)^{\frac{1}{2}} = \{c(x-e_1)(e_2-x)\}^{\frac{1}{2}}$,

Assume $(a+bx-cx^2)^{\frac{1}{2}} = (x-e_1)cy$.

[6] $R\{x, (a+bx)^{\frac{1}{2}}, (a_1+b_1x)^{\frac{1}{2}}\}$;

Assume $a+bx = (a_1+b_1x)y^2$.

[7] $x^{m-1}R\{x^m, x^n, (a+bx^n)^{\frac{p}{q}}\}$.

Assume $a+bx^n = y^q$,

which renders the function rational if $\frac{m}{n}$ is a positive integer.

[8] $x^{m-1}(a+bx^n)^{\frac{p}{q}} . R(x^n)$.

Assume $a+bx^n = y^q$, if $\frac{m}{n}$ is a positive integer,

. . . . . . $ax^{-n}+b = y^q$, if $\frac{m}{n}+\frac{p}{q}$ is a negative integer.

[9] $x^{m-1}R\{x^n, (a+bx^n+cx^{2n})^{\frac{1}{2}}\}$;

may be reduced to [5] by assuming $x^n = y$, if $\frac{m}{n}$ is a positive integer.

[10] $x^{kn-1} R\{x^n, (a+b^2x^{2n})^{\frac{1}{2}}, [bx^n + (a+b^2x^{2n})^{\frac{1}{2}}]\}$;

Assume $y = bx^n + (a+b^2x^{2n})^{\frac{1}{2}}$.

$x^{n-1} R\{x^{mn}, (a+bx^n)^{\frac{p}{q}}, (a+bx^n)^{\frac{r}{s}}, \&c.\}$;

Assume $(a+bx^n) = y^{qs\cdots}$

$x^{n-1} R\left\{x^{mn}, \left(\frac{a+bx^n}{a^1+b^1x^n}\right)^{\frac{p}{q}}, \left(\frac{a+bx^n}{a^1+b^1x^n}\right)^{\frac{r}{s}}, \&c.\right\}$;

Assume $\left(\frac{a+bx^n}{a_1+b_1x^n}\right) = y^{qs\cdots}$

(*L. D. C.* 215—21.)

(6.) The preceding are particular cases of the following more general forms, in which R_1, R_2, &c. denote any rational functions, and R_1^{-1}, R_2^{-1}, &c. the corresponding inverse functions.

[11] $R\{x, R_1^{-1}(x)\}$;

Assume $R_1^{-1}(x) = v$.

[12] $R\{x, R_1^{-1}(x), R_2^{-1}R_1^{-1}(x)$, &c. $R_n^{-1}R_{n-1}^{-1}\ldots R_1^{-1}(x)\}$;

Assume $R_n^{-1}R_{n-1}^{-1}\ldots R_1^{-1}(x) = v$.

[13] $R\{x, R_1^{-1}(x), \overline{R_1^{-1}(x)}|^{\frac{1}{m}}, \overline{R_1^{-1}(x)}|^{\frac{1}{n}}, \&c.\}$;

Assume $R_1^{-1}(x) = v^{mn\cdots}$

[14] $d_xR_2(x).\{R_2(x), R_1^{-1}R_2(x)\}$;

Assume $R_2(x) = v$.

[15] $R\{x, R_1^{-1}R_n(x)\}$;

this formula may be rendered rational if we can determine

$$R_1^{-1}R_n(x) = R_n(x).R_2^{-1}(x).$$

[16] $d_xR_m(x).R\{R_mx, R_1^{-1}R_nR_mx\}$;

this formula may be rendered rational under the same condition as the preceding.

[17] $R\{x, R_1^{-1}(x), \phi_1R_1^{-1}(x), \phi_2R_1^{-1}(x), \&c.\}$

may be reduced to $R\{v, \phi_1(v), \phi_2(v), \&c.\}$ by assuming

$$R_1^{-1}(x) = v.$$

(*Bromhead, Phil. Trans.* 1816.)

Integration of $x^{m-1}(a+bx^n)^p$.

(7.) Let $a+bx^n=X$.

If $\frac{m}{n}$ is a positive integer, for x substitute its value in terms of X obtained from the above equation.

If $\frac{m}{n}+p$ is a negative integer, assume $ax^{-n}+b=Z$, and substitute for x its value in terms of Z.

If neither of these conditions is fulfilled, the indices of x and X may be reduced by some of the following formulæ: from which however no result can be obtained if any of the coefficients become infinite. A result may in these cases be obtained by substituting in the given formulæ for a, b, m, &c. the values which render the coefficients infinite.

$$\begin{aligned}\int_x x^{m-1}X^p &= \frac{x^mX^p}{m}-\frac{pnb}{m}\int_x x^{m+n-1}X^{p-1},\\ &= \frac{x^{m-n}X^{p+1}}{(p+1)nb}-\frac{m-n}{(p+1)nb}\int_x x^{m-n-1}X^{p+1},\\ &= \frac{x^{m-n}X^{p+1}}{(m+np)b}-\frac{(m-n)a}{(m+np)b}\int_x x^{m-n-1}X^{p},\\ &= \frac{x^{m}X^{p}}{m+np}+\frac{pna}{m+np}\int_x x^{m-1}X^{p-1},\\ &= \frac{x^{m}X^{p+1}}{ma}-\frac{(m+n.\overline{p+1})b}{ma}\int_x x^{m+n-1}X^{p},\\ &= -\frac{x^{m}X^{p+1}}{(p+1)na}+\frac{m+n(p+1)}{(p+1)na}\int_x x^{m-1}X^{p+1};\end{aligned}$$

$$\begin{aligned}\int_x \frac{1}{x^{m+1}X^p} &= \frac{1}{(p-1)nax^mX^{p-1}}+\frac{m+n(p-1)}{(p-1)na}\int_x \frac{1}{x^{m+1}X^{p-1}},\\ &= -\frac{1}{max^mX^{p-1}}-\frac{\{m+n(p-1)\}b}{ma}\int_x \frac{1}{x^{m-n+1}X^p},\end{aligned}$$

$$\int_x x^{m-1}X^p = \{A_1x^{m-n} - A_2x^{m-2n} + A_3x^{m-3n} - \&c.$$
$$+ (-1)^{i-1}A_ix^{m-in}\}X^{p+1} + (-1)^iA_i(m-in)\,a\int_x x^{m-in-1}X^p.$$

$$A_1 = \frac{1}{(m+np)b};\ A_2 = \frac{(m-n)a}{\{m+n(p-1)\}b}A_1;\ A_3 = \frac{(m-2n)a}{\{m+n(p-2)\}b}A_2,$$

$$A_i = \frac{\{m-(i-1)n\}a}{\{m+n(p-i+1)\}b}A_{i-1}.$$

$$= \{A_1X^p + A_2X^{p-1} + A_3X^{p-2} + \&c. + A_iX^{p-i+1}\}x^m$$
$$+ A_i(p-i+1)na\int_x x^{m-1}X^{p-i}:$$

$$A_1 = \frac{1}{m+np};\ A_2 = \frac{pna}{m+n\,(p-1)}A_1;\ A_3 = \frac{(p-1)na}{m+n(p-2)}A_2;$$

$$A_i = \frac{(p-i+2)na}{m+(p-i+1)n}A_{i-1}.$$

$$= \{A_1x^m - A_2x^{m+n} + A_3x^{m+2n} - \&c. + (-1)^{i-1}A_ix^{m+(i-1)n}\}X^{p+1}$$
$$+ (-1)^iA_i(m+n.\overline{p+i})b\int_x x^{m+in-1}X^p:$$

$$A_1 = \frac{1}{ma};\ A_2 = \frac{(m+n.\overline{p+1})b}{(m+n)a}A_1;\ A_3 = \frac{(m+n.\overline{p+2})b}{(m+2n)a}A_2;$$

$$A_i = \frac{\{m+n(p+i-1)\}b}{\{m+(i-1)n\}a}A_{i-1}.$$

$$= -\{A_1X^{p+1} + A_2X^{p+2} + A_3X^{p+3} + \ldots\ldots + A_iX^{p+i}\}x^m$$
$$+ A_i(m+n.\overline{p+i})\int_x x^{m-1}X^{p+i}:$$

$$A_1 = \frac{1}{(p+1)na};\ A_2 = \frac{m+n(p+1)}{(p+2)na}A_1;\ A_3 = \frac{m+n(p+2)}{(p+3)na}A_2;$$

$$A_i = \frac{m+n\,(p+i-1)}{(p+i)na}A_{i-1}.$$

(8.) $ax^k + bx^{k+n} = X.$

$$\int_x x^mX^p = \frac{x^{m+1}X^p}{m+pk+1} - \frac{pnb}{m+pk+1}\int_x x^{m+k+n}X^{p-1}.$$

$$\int_x x^mX^p = \frac{x^{m-k-n+1}X^{p+1}}{(m+p.\overline{k+n}+1)b} - \frac{(m+pk-n+1)a}{(m+p.\overline{k+n}+1)b}\int_x x^{m-n}X^p.$$

$$\int_x \frac{X^p}{x^m} = -\frac{X^p}{(m-p\cdot\overline{k+n}-1)x^{m-1}} - \frac{pna}{m-p(k+n)-1}\int_x \frac{X^{p-1}}{x^{m-k}}.$$

$$\int_x \frac{X^p}{x^m} = -\frac{X^{p+1}}{(m-pk-1)ax^{m+k-1}} - \frac{(m-n-p.\overline{k+n}-1)b}{(m-pk-1)a}\int_x \frac{X^p}{x^{m-n}}.$$

$$\int_x \frac{x^m}{X^p} = -\frac{x^{m-k-n+1}}{(p-1)nbX^{p-1}} + \frac{m-pk-n+1}{(p-1)nb}\int_x \frac{x^{m-k-n}}{X^{p-1}}.$$

$$\int_x \frac{x^m}{X^p} = \frac{x^{m-k+1}}{(p-1)naX^{p-1}} - \frac{m+n-p(k+n)+1}{(p-1)na}\int_x \frac{x^{m-k}}{X^{p-1}}.$$

(9.) $a+bx=X.$

$$\int_x \frac{x^m}{X} = \frac{x^m}{mb} - \frac{a}{b}\int_x \frac{x^{m-1}}{X}.$$

$$\int_x \frac{x^m}{X^2} = \frac{x^m}{(m-1)bX} - \frac{ma}{(m-1)b}\int_x \frac{x^{m-1}}{X^2}.$$

$$\int_x \frac{1}{x^mX} = -\frac{1}{(m-1)ax^{m-1}} - \frac{b}{a}\int_x \frac{1}{x^{m-1}X}.$$

$$\int_x \frac{1}{x^mX^2} = -\frac{1}{(m-1)ax^{m-1}X} - \frac{mb}{(m-1)a}\int_x \frac{1}{x^{m-1}X^2}.$$

(10.) $a+bx^2=X.$

$$\int_x \frac{1}{X^p} = \frac{x}{(p-1)2aX^{p-1}} + \frac{2p-3}{(p-1)2a}\int_x \frac{1}{X^{p-1}}.$$

$$\int_x \frac{x^m}{X} = \frac{x^{m-1}}{(m-1)b} - \frac{a}{b}\int_x \frac{x^{m-2}}{X}.$$

$$\int_x \frac{x^m}{X^2} = \frac{x^{m-1}}{(m-3)bX} - \frac{(m-1)a}{(m-3)b}\int_x \frac{x^{m-2}}{X^2}.$$

$$\int_x \frac{1}{x^mX} = -\frac{1}{(m-1)ax^{m-1}} - \frac{b}{a}\int_x \frac{1}{x^{m-2}X}.$$

$$\int_x \frac{1}{x^mX^2} = -\frac{1}{(m-1)ax^{m-1}X} - \frac{(m+1)b}{(m-1)a}\int_x \frac{1}{x^{m-2}X^2}.$$

$$\int_x x^m X^{\frac{n}{2}} = \frac{x^{m+1} X^{\frac{n}{2}}}{m+1} - \frac{nb}{m+1} \int_x x^{m+2} X^{\frac{n}{2}-1}.$$

$$\int_x \frac{x^m}{X^{\frac{n}{2}}} = \frac{x^{m-1}}{(n-2) b X^{\frac{n}{2}-1}} + \frac{m-1}{(n-2)b} \int_x \frac{x^{m-2}}{X^{\frac{n}{2}-1}}.$$

$$\int_x x^m X^{\frac{n}{2}} = \frac{x^{m-1} X^{\frac{n}{2}+1}}{(m+n+1)b} - \frac{(m-1)a}{(m+n+1)b} \int_x x^{m-2} X^{\frac{n}{2}}.$$

$$\int_x x^m X^{\frac{n}{2}} = \frac{x^{m+1} X^{\frac{n}{2}}}{m+n+1} + \frac{na}{m+n+1} \int_x x^m X^{\frac{n}{2}-1}.$$

$$\int_x \frac{X^{\frac{n}{2}}}{x^m} = -\frac{X^{\frac{n}{2}+1}}{(m-1) a x^{m-1}} - \frac{(m-n-3)b}{(m-1)a} \int_x \frac{X^{\frac{n}{2}}}{x^{m-2}}.$$

$$\int_x \frac{x^m}{X^{\frac{n}{2}}} = \frac{x^{m+1}}{(n-2) a X^{\frac{n}{2}-1}} - \frac{m-n+3}{(n-2) a} \int_x \frac{x^m}{X^{\frac{n}{2}-1}}.$$

(11.) $ax + bx^2 = X.$

$$\int_x x^m X^{\frac{n}{2}} = \frac{2x^{m+1} X^{\frac{n}{2}}}{2m+n+2} - \frac{nb}{2m+n+2} \int_x x^{m+2} X^{\frac{n}{2}-1},$$

$$\int_x \frac{x^m}{X^{\frac{n}{2}}} = -\frac{2x^{m-1}}{(n-2) b X^{\frac{n}{2}-1}} + \frac{2m-n}{(n-2)b} \int_x \frac{x^{m-2}}{X^{\frac{n}{2}-1}}.$$

$$\int_x x^m X^{\frac{n}{2}} = \frac{x^{m-1} X^{\frac{n}{2}-1}}{(m+n+1)b} - \frac{(2m+n)a}{(m+n+1)2b} \int_x x^{m-1} X^{\frac{n}{2}}.$$

$$\int_x \frac{X^{\frac{n}{2}}}{x^m} = -\frac{X^{\frac{n}{2}}}{(m-n-1) x^{m-1}} - \frac{na}{2(m-n-1)} \int_x \frac{X^{\frac{n}{2}-1}}{x^{m-1}}.$$

$$\int_x \frac{X^{\frac{n}{2}}}{x^m} = -\frac{2X^{\frac{n}{2}+1}}{(2m-n-2) a x^m} - \frac{(m-n-2)2b}{(2m-n-2)a} \int_x \frac{X^{\frac{n}{2}}}{x^{m-1}}.$$

$$\int_x \frac{x^m}{X^{\frac{n}{2}}} = \frac{2x^m}{(n-2) a X^{\frac{n}{2}-1}} - \frac{2(m-n+2)}{(n-2)a} \int_x \frac{x^{m-1}}{X^{\frac{n}{2}-1}}.$$

(12.) $a+bx^n+cx^{2n}=X.$

$$\int_x x^{m-1}X^p=\frac{x^mX^p}{m}-\frac{pnb}{m}\int_x x^{m+n-1}X^{p-1}-\frac{2pnc}{m}\int_x x^{m+2n-1}X^{p-1}:$$

$$=\frac{x^{m-2n}X^{p+1}}{(m+2pn)c}-\frac{(m-2n)a}{(m+2pn)c}\int_x x^{m-2n-1}X^p$$

$$-\frac{\{m+n(p-1)\}b}{(m+2pn)c}\int_x x^{m-n-1}X^p:$$

$$=\frac{x^nX^p}{m+2pn}+\frac{2pna}{m+2pn}\int_x x^{m-1}X^{p-1}+\frac{pnb}{m+2pn}\int_x x^{m+n-1}X^{p-1}:$$

$$=\frac{x^mX^{p+1}}{ma}-\frac{\{m+n(p+1)\}b}{ma}\int_x x^{m+n-1}X^p$$

$$-\frac{\{m+2n(p+1)\}c}{ma}\int_x x^{m+2n-1}X^p.$$

(13.) $a+bx^2+cx^4=X.$

$$\int_x\frac{1}{X^p}=\frac{bcx^3+(b^2-2ac)x}{(p-1)(b^2-4ac)2aX^{p-1}}+\frac{(4p-5)bc}{(p-1)(b^2-4ac)2a}\int_x\frac{x^2}{X^{p-1}}$$

$$+\frac{2(p-1)(b^2-4ac)+2ac-b^2}{2(p-1)(b^2-4ac)a}\int_x\frac{1}{X^{p-1}}.$$

$$\int_x\frac{x^m}{X^p}=\frac{bcx^{m+3}+(b^2-2ac)x^{m+1}}{2a(p-1)(b^2-4ac)X^{p-1}}+\frac{(4p-m-5)bc}{2(p-1)(b^2-4ac)a}\int_x\frac{x^{m+2}}{X^{p-1}}$$

$$+\frac{2(p-1)(b^2-4ac)+(m+1)(2ac-b^2)}{2a(p-1)(b^2-4ac)}\int_x\frac{x^m}{X^{p-1}},$$

$$=\frac{x^{m-3}}{(m-4p+1)cX^{p-1}}-\frac{(m-3)a}{(m-4p+1)c}\int_x\frac{x^{m-4}}{X^p}$$

$$-\frac{(m-2p-1)b}{(m-4p+1)c}\int_x\frac{x^{m-2}}{X^p}.$$

$$\int_x\frac{1}{x^mX^p}=-\frac{1}{(m-1)ax^{m-1}X^{p-1}}-\frac{(m+2p-3)b}{(m-1)a}\int_x\frac{1}{x^{m-2}X^p}$$

$$-\frac{(m+4p-5)c}{(m-1)a}\int_x\frac{1}{x^{m-4}X^p}.$$

(14.) $a+bx+cx^2=X$.

$$\int_x \frac{1}{X^p} = \frac{2cx+b}{(p-1)(4ac-b^2)X^{p-1}} + \frac{(2p-3)2c}{(p-1)(4ac-b^2)}\int_x \frac{1}{X^{p-1}}.$$

$$\int_x \frac{x^m}{X} = \frac{x^{m-1}}{(m-1)c} - \frac{a}{b}\int_x \frac{x^{m-2}}{X} - \frac{b}{c}\int_x \frac{x^{m-1}}{X}.$$

$$\int_x \frac{x^m}{X^2} = \frac{x^{m-1}}{(m-3)cX} - \frac{(m-1)a}{(m-3)c}\int_x \frac{x^{m-2}}{X^2} - \frac{(m-2)b}{(m-3)c}\int_x \frac{x^{m-1}}{X^2}.$$

$$\int_x \frac{1}{x^mX} = -\frac{1}{(m-1)ax^{m-1}} - \frac{b}{a}\int_x \frac{1}{x^{m-1}X} - \frac{c}{a}\int_x \frac{1}{x^{m-2}X}.$$

$$\int_x \frac{1}{x^mX^2} = -\frac{1}{(m-1)ax^{m-1}X} - \frac{mb}{(m-1)a}\int_x \frac{1}{x^{m-1}X^2}$$
$$- \frac{(m+1)c}{(m-1)a}\int_x \frac{1}{x^{m-2}X^2}.$$

$$\int_x x^mX^p = \frac{x^{m+1}X^p}{m+1} - \frac{pb}{m+1}\int_x x^{m+1}X^{p-1} - \frac{2pc}{m+1}\int_x x^{m+2}X^{p-1}.$$

$$\int_x \frac{x^m}{X^p} = \frac{x^{m-1}}{(m-2p+1)cX^{p-1}} - \frac{(m-1)a}{(m-2p+1)c}\int_x \frac{x^{m-2}}{X^p}$$
$$- \frac{(m-p)b}{(m-2p+1)c}\int_x \frac{x^{m-1}}{X^p}.$$

$$\int_x \frac{X^p}{x^m} = -\frac{X^p}{(m-1)x^{m-1}} + \frac{pb}{m-1}\int_x \frac{X^{p-1}}{x^{m-1}} + \frac{2pc}{m-1}\int_x \frac{X^{p-1}}{x^{m-2}}.$$

$$\int_x \frac{X^p}{x^m} = -\frac{X^p}{(m-2p-1)x^{m-1}} - \frac{2pa}{m-2p-1}\int_x \frac{X^{p-1}}{x^m}$$
$$- \frac{pb}{m-2p-1}\int_x \frac{X^{p-1}}{x^{m-1}}.$$

$$\int_x \frac{1}{x^mX^p} = -\frac{1}{(m-1)ax^{m-1}X^{p-1}} - \frac{(m+p-2)b}{(m-1)a}\int_x \frac{1}{x^{m-1}X^p}$$
$$- \frac{(m+2p-3)c}{(m-1)a}\int_x \frac{1}{x^{m-2}X^p}.$$

(*Hirsch*, *Int. Tab.*; *L. C. D.* 391—404.)

(15.) *Development of* $\int_x x^{m-1}(a+bx^n)^p$ *by series.*

$$\int_x x^{m-1}(a+bx^n)^p =$$

$$\begin{aligned} a^p x^m \Big\{ \frac{1}{m} &+ \frac{b}{a}\cdot\frac{px^n}{m+n} + \frac{b^2}{1.2.a^2}\cdot\frac{p(p-1)x^{2n}}{m+2n} \\ &+ \frac{b^3}{1.2.3.a^3}\cdot\frac{p(p-1)(p-2)x^{3n}}{m+3n} + \&\text{c.}\Big\} \end{aligned}$$

$$b^p x^{m+np}\left\{\frac{1}{m+np} + \frac{a}{b}\cdot\frac{px^{-n}}{m+n(p-1)} + \frac{a^2}{1.2.b^2}\cdot\frac{p(p-1)x^{-2n}}{m+n(p-2)} + \&\text{c.}\right\}$$

$$\begin{aligned} x^m X^{p+1}\Big\{ \frac{x^{-n}}{(m+np)b} &- \frac{(m-n)ax^{-2n}}{(m+np)(m+n.\overline{p-1})b^2} \\ &+ \frac{(m-n)(m-2n)a^2x^{-3n}}{(m+np)(m+n.\overline{p-1})(m+n.\overline{p-2})b^3} - \&\text{c.}\Big\} \end{aligned}$$

$$\begin{aligned} x^m X^{p+1}\Big\{ \frac{1}{ma} &- \frac{(m+n.\overline{p+1})b}{m(m+n)a^2}x^n \\ &+ \frac{(m+n.\overline{p+1})(m+n.\overline{p+2})b^2}{m(m+n)(m+2n)a^3}x^{2n} - \&\text{c.}\Big\} \end{aligned}$$

$$\begin{aligned} -x^m X^{p+1}\Big\{ \frac{1}{(p+1)na} &+ \frac{m+n(p+1)}{(p+1)(p+2)n^2a^2}X \\ &+ \frac{(m+n.\overline{p+1})(m+n.\overline{p+2})}{(p+1)(p+2)(p+3)n^3a^3}X^2 + \&\text{c.}\Big\} \end{aligned}$$

$$\begin{aligned} x^m X^p\Big\{ \frac{1}{m+np} &+ \frac{pna}{(m+np)(m+n.\overline{p-1})}X^{-1} \\ &+ \frac{p(p-1)n^2a^2}{(m+np)\ldots(m+n.\overline{p-2})}X^{-2} + \&\text{c.}\Big\} \end{aligned}$$

$$x^m X^p\left\{\frac{1}{m} - \frac{pnb}{m(m+n)}\cdot\frac{x^n}{X} + \frac{p(p-1)n^2b^2}{m(m+n)(m+2n)}\cdot\left(\frac{x^n}{X}\right)^2 - \&\text{c.}\right\}$$

$$\begin{aligned} x^{m-n} X^{p+1}\Big\{ \frac{1}{(p+1)nb} &- \frac{m-n}{(p+1)(p+2)n^2b^2}\cdot\frac{X}{x^n} \\ &+ \frac{(m-n)(m-2n)}{(p+1)(p+2)(p+3)n^3b^3}\cdot\left(\frac{X}{x^n}\right)^2 - \&\text{c.}\Big\} \end{aligned}$$

(*Tr. L.* 175—80; *Hirsch, Int. Tab.*)

(16.) *Reduction of* $\dfrac{R(x)}{(a+a_1x+a_2x^2+a_3x^3+a_4x^4)^{\frac{1}{2}}}$.

Assume $x=\dfrac{b+cy}{1+y}$, then the denominator, which may be represented by X, becomes

$$\frac{(e+e_1y+e_2y^2+e_3y^3+e_4y^4)^{\frac{1}{2}}}{(1+y)^2},$$

in which e, e_1, &c. are functions of a, a_1, &c. b, and c.

Let the values of b and c be determined from the equations

$$e_1=0, \quad e_3=0;$$

then $\int_x \dfrac{R(x)}{X}$ becomes $\int_y \dfrac{R(y)}{(e+e_2y^2+e_4y^4)^{\frac{1}{2}}}$, or $\int_y \dfrac{R(y)}{Y}$:

this again may be separated into

$$\int_y \frac{R_1(y^2)}{Y}+\int_y \frac{y.R_2(y^2)}{Y}.$$

Let $y^2=z$, then $\int_y \dfrac{y.R_2(y^2)}{Y}$ becomes $\int_z \dfrac{R(z)}{2(e+e_2z+e_4z^2)^{\frac{1}{2}}}$, which may be integrated by preceding methods.

If R_1 is an integral function, each term of $\int_y \dfrac{R_1(y^2)}{Y}$ may be integrated, or reduced by preceding methods to the form

$$\int_y \frac{1}{Y} \text{ or } \int_y \frac{y^2}{Y}.$$

If $R_1(y^2)$ is a fraction, it may be decomposed, and each separate fraction may be integrated, or reduced to the form

$$\int_y \frac{1}{(y^2+a)Y}.$$

$R(x).\overline{\dfrac{a+a_1x+a_2x^2}{b+b_1x+b_2x^2}}\Big|^{\frac{1}{2}}$ may be reduced to the form $\dfrac{R(x)}{X}$ by multiplying the numerator and denominator of the fraction by the denominator.

$\int_x \frac{R(x)}{(a + a_1x^2 + a_2x^4 + a_3x^6)^{\frac{1}{2}}}$ may be separated into two parts ⫲ $\int_x \frac{R(x)}{X}$ by separating $R(x)$ into $R_1(x^2)$, and $x.R_2(x^2)$, and putting $x^2 = y$.

$\int_x \frac{R(x)}{(a + a_1x^2 + a_2x^4)^{\frac{1}{4}}}$ may be separated into two parts

$$\int_x \frac{R_1(x^2)}{(a + a_1x^2 + a_2x^4)^{\frac{1}{4}}}, \text{ and } \int_x \frac{x.R_2(x^2)}{(a + a_1x^2 + a_2x^4)^{\frac{1}{4}}}:$$

to reduce the first of these, assume

$$(a + a_1x^2 + a_2x^4)^{\frac{1}{4}} = xy;$$

to reduce the second, assume

$$(a + a_1x^2 + a_2x^4)^{\frac{1}{4}} = z;$$

then by substituting for x its value, we obtain respectively

$$\int_y \frac{y^2.R_3(y^4)}{(a_1{}^2 - 4aa_2 + 4ay^4)^{\frac{1}{2}}}, \text{ and } \int_z \frac{z^2.R_4(z^4)}{(a_1{}^2 - 4aa_2 + 4a_2z^4)^{\frac{1}{2}}}.$$

$\int_x R(x).\overline{a + a_1x^2 + a_2x^4}\,\big|^{\frac{1}{4}}$ and $\int_x R(x).\overline{a + a_1x^2 + a_2x^4}\,\big|^{\pm\frac{3}{4}}$ may be transformed in the same manner as the preceding formula.

To reduce $\int_x R(x).\overline{a + a_1x + a_2x^2 + a_3x^3}\,\big|^{\pm\frac{1}{3}}$ assume

$$\overline{a + a_1x + a_2x^2 + a_3x^3}\,\big|^{\frac{1}{3}} = a^{\frac{1}{3}} + xz, \text{ or} = a_3{}^{\frac{1}{3}}x + z,$$

by either of which assumptions, the term a_3x^3 will disappear from the radical.

To reduce $\int_x \frac{R(x)}{(a + a_1x^2 + a_2x^4 + a_1x^6 + ax^8)^{\frac{1}{2}}}$, separate $R(x)$ into $R_1(x^2)$ and $x.R_2(x^2)$, and put $y = x^2$: the part corresponding to $R_1(x^2)$ may be further reduced by assuming

$$y + \frac{1}{y} = z.$$

(*L. C. D.* 406—11.)

(17.) *Integration of* $\frac{R(x)}{(a+a_1x^2+a_2x^4)^{\frac{1}{2}}}$.

Let $a+a_1x^2+a_2x^4=(b_1+c_1x^2)(b_2+c_2x^2)$,

$$=b_1.b_2(1+e_1^2.x^2)(1+e_2^2.x^2).$$

[1] Let the quantities e_1, e_2 be very unequal, and $e_1>e_2$; assume $e_1^2x^2=y^2$, then

$$\int_x\frac{R(x)}{(a+a_1x^2+a_2x^4)^{\frac{1}{2}}}\text{ becomes }\frac{b_1.b_2}{e_1}\int_y\frac{R(y)}{\left\{(1+y^2)\left(1+\frac{e_2^2}{e_1^2}y^2\right)\right\}^{\frac{1}{2}}},$$

and since $\left(1+\frac{e_2^2}{e_1^2}y^2\right)^{-\frac{1}{2}}$ may be expanded by the binomial theorem in a series which converges rapidly, if $\frac{e_2}{e_1}$ is small, the required integral will depend on $\int_y\frac{R(y)}{(1-ay^2).\overline{1+y^2}|^{\frac{1}{2}}}$.

[2] If e_1 and e_2 are nearly equal, put the given function under the form

$$\frac{1}{e_1.e_2}.\frac{R(x)}{\left\{\left(\frac{1}{e_1^2}+x^2\right)\left(\frac{1}{e_2^2}+x^2\right)\right\}^{\frac{1}{2}}},$$

and assume $\frac{1}{e_1^2}=r^2-k$, and $\frac{1}{e_2^2}=r^2+k$, then by substituting these values we obtain $\int_x\frac{R(x)}{\{(r^2+x^2)^2-k^2\}^{\frac{1}{2}}}$

$$=\int_x R(x).\left\{\frac{1}{r^2+x^2}+\frac{1}{2}\cdot\frac{k^2}{(r^2+x^2)^3}+\frac{1.3}{2.4}\cdot\frac{k^4}{(r^2+x^2)^5}+\&c.\right\},$$

a series which converges rapidly if r^2 is positive, and k small; the required integral is thus made to depend on

$$\int_x\frac{R(x)}{(r^2+x^2)-a}.$$

Lagrange's Method. Assume $y=\frac{x}{b_1}.\overline{\frac{b_1+c_1x^2}{b_2+c_2x^2}}\Big|^{\frac{1}{2}}$, then

$$\int_x\frac{R(x)}{(a+a_1a^2+a_2x^4)^{\frac{1}{2}}}\text{ becomes }\int_y R(y^2)+\int_y\frac{R_1(y^2)}{\overline{(1\pm p^2y^2)(1\pm q^2y^2)}|^{\frac{1}{2}}},$$

where p^2y^2 and q^2y^2 are either both positive or both negative, and

$$p = (\pm c_1 b_2)^{\frac{1}{2}} + \{\pm(c_1 b_2 - b_1 c_2)\}^{\frac{1}{2}},$$

$$q = (\pm c_1 b_2)^{\frac{1}{2}} - \{\pm(c_1 b_2 - b_1 c_2)\}^{\frac{1}{2}},$$

$$\text{or} \quad q = \{\pm(c_1 b_2 - b_1 c_2)\}^{\frac{1}{2}} - (\pm c_1 b_2);$$

+ or − according as c_1 and b_2 have the same or different signs: and the second value of q must be taken, if $c_1 b_2$ and $b_1 c_2$ have different signs.

The same transformation may be repeated by assuming

$$y_1 = \frac{yQ}{1 \pm q^2 y^2}, \quad p_1 = p + (p^2 - q^2)^{\frac{1}{2}}, \quad q_1 = p - (p^2 - q^2)^{\frac{1}{2}};$$

$$y_2 = \frac{y_1 Q_1}{1 \pm q_1^2 y_1^2}, \quad p_2 = p_1 + (p_1^2 - q_1^2)^{\frac{1}{2}}, \quad q_2 = p_1 - (p_1^2 - q_1^2)^{\frac{1}{2}};$$

$$\&\text{c.} \qquad\qquad \&\text{c.}$$

where Q_m represents $\{(1 \pm p_m^2 \cdot y_m^2)(1 \pm q_m^2 \cdot y_m^2)\}^{\frac{1}{2}}$:
or the values of p_1, q_1, &c. may be thus calculated; let

$$p + q = r, \qquad p - q = s,$$

$$p_1 + q_1 = r_1, \qquad p_1 - q_1 = s_1,$$

$$\&\text{c.} \qquad\qquad \&\text{c.}$$

then
$$r_1 = r + s, \quad \text{and} \quad s_1 = 2(rs)^{\frac{1}{2}}.$$

$$\&\text{c.} \qquad\qquad \&\text{c.}$$

by this process we obtain a series of the form

$$\int_y R(y^2) + \int_{y_1} R(y_1^2) + \&\text{c.}$$

$$+ \int_{y_n} R(y_n^2) + \int_{y_n} \frac{R_1(y_n^2)}{\overline{(1 \pm p_n^2 \cdot y_n^2)(1 \pm q_n^2 \cdot y_n^2)}|^{\frac{1}{2}}},$$

in which, since $p \sim q$ continually increases, $p_n \sim q_n$ may be made as large as we please.

Similarly by assuming

$$p = p_{-1} + (p_{-1}^2 - q_{-1}^2)^{\frac{1}{2}}, \qquad q = p_{-1} - (p_{-1}^2 - q_{-1}^2)^{\frac{1}{2}},$$

$$\&\text{c.} \qquad\qquad \&\text{c.}$$

or
$$p_{-1} = \tfrac{1}{2}(p + q), \qquad q_{-1} = (pq)^{\frac{1}{2}},$$

$$\&\text{c.} \qquad\qquad \&\text{c.}$$

we may obtain a series of the form

$$\int_y R(y^2) + \int_{y_{-1}} R(y^2_{-1}) + \&c.$$

$$+ \int_{y_{-n}} R(y^2_{-n}) + \int_{y_{-n}} \frac{R_1(y^2_{-n})}{\overline{(1 \pm p^2_{-n}.y^2_{-n})(1 \pm q^2_{-n}.y^2_{-n})}\,|^{\frac{1}{2}}},$$

in which $p_{-n} \sim q_{-n}$ may be made as small as we please: in either case, the value of the last term may be expressed by a converging series, by the methods already given.

(*L. C. D.* 423—6.)

INTEGRATION OF TRANSCENDENTAL FUNCTIONS.

(18.) *Exponential and logarithmic functions.*

$$\int_x \frac{1}{x} = \log_e x + const.$$

$$\int_x a^x = \frac{a^x}{\log_e a} + const.$$

$$\int_x a^x.u = \frac{a^x.u}{\log_e.a} - \frac{a^x.d_x u}{(\log_e a)^2} + \frac{a^x.d_x^2 u}{(\log_e a)^3} - \&c.$$

$$\int_x x^n a^x = \frac{a^x}{\log_e a}\left\{x^n - \frac{n x^{n-1}}{\log_e a} + \frac{n(n-1)x^{n-2}}{(\log_e a)^2} - \&c.\right\}$$

$$\int_x \frac{a^x}{x} = \log_e x + \frac{x.\log_e a}{1.1} + \frac{(x.\log_e a)^2}{1.2.2} + \frac{(x.\log_e a)^3}{1.2.3.3} + \&c.$$

$$\int_x x^{m+nx} = x^{m+1}\left\{\left(\frac{1}{m+1} - \frac{nx}{(m+2)^2} + \frac{(nx)^2}{(m+3)^3} - \&c.\right)\right.$$
$$+ \frac{nx.\log_e x}{1}\left(\frac{1}{m+2} - \frac{nx}{(m+3)^2} + \frac{(nx)^2}{(m+4)^3} - \&c.\right)$$
$$+ \frac{(nx.\log_e x)^2}{1.2}\left(\frac{1}{m+3} - \frac{nx}{(m+4)^2} + \frac{(nx)^2}{(m+5)^3} - \&c.\right)$$
$$\left.+ \&c.\right\} \; ^{\alpha}$$

$^{\alpha}$ $\int_x x^{m+nx} = x^{m+1}.\overset{\infty}{\mathrm{S}}_r \frac{(nx.\log_e x)^{r-1}}{\underline{|r-1}}.\overset{\infty}{\mathrm{S}}_s \frac{(-nx)^{s-1}}{(m+r+s-1)^s}.$

$$\int_x u\log_\varepsilon v = \log_\varepsilon v\,.\int_x u - \int_x \frac{d_x v}{v}\int_x u.$$

$$\int_x u\log_\varepsilon x = \log_\varepsilon x\int_x u - \int_x \frac{\int_x u}{x}.$$

$$\int_x u\,(\log_\varepsilon x)^n = (\log_\varepsilon x)^n\int_x u - n\,(\log_\varepsilon x)^{n-1}\int_x \frac{1}{x}\int_x u$$
$$+\,n\,(n-1)\,(\log_\varepsilon x)^{n-2}\int_x \frac{1}{x}\int_x \frac{1}{x}\int_x u - \&c.$$

$$\int_x \frac{u}{(\log_\varepsilon x)^n} = -\frac{ux}{(n-1)\,(\log_\varepsilon x)^{n-1}} - \frac{d_x ux}{(n-1)\,(n-2)\,(\log_\varepsilon x)^{n-2}}$$
$$-\frac{d_x(x\,.\,d_x ux)}{(n-1)\,(n-2)\,(n-3)\,(\log_\varepsilon x)^{n-3}} - \&c.$$

$$\int_x x^m(\log_\varepsilon x)^n = \frac{x^{m+1}}{m+1}\left\{(\log_\varepsilon x)^n - \frac{n}{m+1}\,(\log_\varepsilon x)^{n-1}\right.$$
$$\left.+\,\frac{n\,(n-1)}{(m+1)^2}\,(\log_\varepsilon x)^{n-2} - \&c.\right\} \quad \beta$$

$$\int_x \frac{x^n}{(\log_\varepsilon x)^n} = -x^{m+1}\left\{\frac{1}{(n-1)(\log_\varepsilon x)^{n-1}} + \frac{m+1}{(n-1)(n-2)(\log_\varepsilon x)^{n-2}}\right.$$
$$\left.+\,\frac{(m+1)^2}{(n-1)(n-2)(n-3)(\log_\varepsilon x)^{n-3}} + \&c.\right\} \quad \gamma$$

$$\beta\ \int_x x^m(\log_\varepsilon x)^n = \frac{x^{m+1}}{m+1}\overset{r}{\mathrm{S}}_s\left(\frac{-1}{m+1}\right)^{s-1}\cdot\lfloor\underline{n}_{\,s-1}\,.\,(\log_\varepsilon x)^{n-s+1}$$
$$-\,\frac{\lfloor\underline{n}_{\,r}}{(m+1)^r}\int_x x^m(\log_\varepsilon x)^{n-r}.$$

$$\gamma\ \int_x \frac{x^m}{(\log_\varepsilon x)^n} = -\,x^{m+1}\,\overset{r}{\mathrm{S}}_s\,\frac{(m+1)^{s-1}}{\lfloor\underline{n-1}_{\,s}\,.\,(\log_\varepsilon x)^{n-s}}$$
$$+\,\frac{(m+1)^r}{\lfloor\underline{n-1}_{\,r}}\int_x \frac{x^m}{(\log_\varepsilon x)^{n-r}}.$$

$$\int_x \frac{1}{\log_\epsilon x} = \log_\epsilon^2 x + \frac{\log_\epsilon x}{1} + \frac{(\log_\epsilon x)^2}{1.2.2} + \frac{(\log_\epsilon x)^3}{1.2.3.3} + \&c.\ ^{\alpha}$$

(*L. C. D.* 427—37; *Tr. L.* 181—91; *Hirsch, Int. Tab.*)

(19.) *Integration of circular functions.*

$$\int_x \cos x = \sin x + const.$$

$$\int_x \sin x = -\cos x + const.$$

$$\int_x \frac{1}{(\cos x)^2} = \int_x (\sec x)^2 = \tan x + const.$$

$$\int_x \frac{1}{(\sin x)^2} = \int_x (\operatorname{cosec} x)^2 = -\cot x + const.$$

$$\int_x \frac{\sin x}{(\cos x)^2} = \int_x \tan x \,.\, \sec x = \sec x + const.$$

$$\int_x \frac{\cos x}{(\sin x)^2} = \int_x \cot x \,.\, \operatorname{cosec} x = -\operatorname{cosec} x + const.$$

$$\int_x \sin x \,.\, \cos x = \tfrac{1}{2}(\sin x)^2 + const.$$

$$\int_x \frac{1}{\sin x} = \int_x \operatorname{cosec} x = \log \tan \tfrac{1}{2}x + const.$$

$$\int_x \frac{1}{\cos x} = \int_x \sec x = \log \cot (45^\circ - \tfrac{1}{2}x) + const.$$

$$\int_x \frac{\sin x}{\cos x} = \int_x \tan x = -\log \cos x + const.$$

$$\int_x \frac{\cos x}{\sin x} = \int_x \cot x = \log \sin x + const.$$

$$\int_x \frac{1}{\sin x \,.\, \cos x} = \log \tan x + const.$$

$^{\alpha}$ $\displaystyle\int_x \frac{1}{\log_\epsilon x} = \log_\epsilon^2 x + \overset{\infty}{\mathrm{S}}_m \frac{(\log_\epsilon x)^m}{m\,.\lfloor\underline{m}}.$

(20) *Formulæ for the reduction of* $\int_x (\sin x)^m . (\cos x)^n$.

$$\int_x (\sin x)^m . (\cos x)^n$$

$$= \frac{(\sin x)^{m+1} . (\cos x)^{n-1}}{m+1} + \frac{n-1}{m+1} \int_x (\sin x)^{m+2} . (\cos x)^{n-2};$$

$$= - \frac{(\sin x)^{m-1} . (\cos x)^{n+1}}{n+1} + \frac{m-1}{n+1} \int_x (\sin x)^{m-2} . (\cos x)^{n+2};$$

$$= - \frac{(\sin x)^{m-1} . (\cos x)^{n+1}}{m+n} + \frac{m-1}{m+n} \int_x (\sin x)^{m-2} . (\cos x)^{n};$$

$$= \frac{(\sin x)^{m+1} . (\cos x)^{n-1}}{m+n} + \frac{n-1}{m+n} \int_x (\sin x)^{m} . (\cos x)^{n-2};$$

$$= \frac{(\sin x)^{m+1} . (\cos x)^{n+1}}{n+1} + \frac{m+n+2}{m+1} \int_x (\sin x)^{m+2} . (\cos x)^{n};$$

$$= - \frac{(\sin x)^{m+1} . (\cos x)^{n+1}}{n+1} + \frac{m+n+2}{n+1} \int_x (\sin x)^{m} . (\cos x)^{n+2}.$$

$$\int_x (\sin x)^n = - \frac{(\sin x)^{n-1} . \cos x}{n} + \frac{n-1}{n} \int_x (\sin x)^{n-2}.$$

$$\int_x (\cos x)^n = \frac{\sin x (\cos x)^{n-1}}{n} + \frac{n-1}{n} \int_x (\cos x)^{n-2}.$$

$$\int_x \frac{1}{(\sin x)^n} = - \frac{\cos x}{(n-1)(\sin x)^{n-1}} + \frac{n-2}{n-1} \int_x \frac{1}{(\sin x)^{n-2}}.$$

$$\int_x \frac{1}{(\cos x)^n} = \frac{\sin x}{(n-1)(\cos x)^{n-1}} + \frac{n-2}{n-1} \int_x \frac{1}{(\cos x)^{n-2}}.$$ [α]

(*L. D. C.* 250—8; *L. C. D.* 441—8.)

[α] $\int_x (\sin x)^n = - \cos x . \overset{\frac{1}{2}n}{\mathrm{S}}_m \frac{\underline{\left| n-1 \right.}_{m-1,-2}}{\underline{\left| n \right.}_{m,-2}} (\sin x)^{n-2m+1} + \frac{\underline{\left| 1 \right.}_{\frac{1}{2}n,2}}{\underline{\left| 2 \right.}_{\frac{1}{2}n,2}} x$, ($n$ even);

$= - \cos x . \overset{\frac{1}{2}(n+1)}{\mathrm{S}}_m \frac{\underline{\left| n-1 \right.}_{m-1,-2}}{\underline{\left| n \right.}_{m,-2}} (\sin x)^{n-2m+1}$, ($n$ odd).

$\int_x (\cos x)^n$

(21.) $\int_x x^n.\sin x =$

$$-\cos x\,\{x^n - n(n-1)\,x^{n-2} + n\ldots(n-3)\,x^{n-4} - \&c.\}$$

$$+\sin x\,\{nx^{n-1} - n(n-1)(n-2)\,x^{n-3} + n\ldots(n-4)\,x^{n-5} - \&c.\}.$$

$$\int_x x^n.\cos x = \sin x\,\{x^n - n(n-1)\,x^{n-2} + n\ldots(n-3)\,x^{n-4} - \&c.\}$$

$$+\cos x\,\{nx^{n-1} - n(n-1)(n-2)\,x^{n-3} + n\ldots(n-4)\,x^{n-5} - \&c.\}.$$

$$\int_x \epsilon^{ax}(\sin x)^n = \frac{\epsilon^{ax}(\sin x)^{n-1}.(a\sin x - n\cos x)}{a^2+n^2}$$

$$+\frac{n(n-1)}{a^2+n^2}\int_x \epsilon^{ax}(\sin x)^{n-2}.$$

$$\int_x (\cos x)^n = \sin x.\overset{\frac{1}{2}n}{\mathrm{S}}_m \frac{\underline{|n-1}_{\,m-1,-2}}{\underline{|n}_{\,m,-2}}(\cos x)^{n-2m+1} + \frac{\underline{|1}_{\,\frac{1}{2}n,2}}{\underline{|2}_{\,\frac{1}{2}n,2}}\,x,\ (n \text{ even}),$$

$$= \sin x.\overset{\frac{1}{2}(n+1)}{\mathrm{S}}_m \frac{\underline{|n-1}_{\,m-1,-2}}{\underline{|n}_{\,m,-2}}(\cos x)^{n-2m+1},\ (n \text{ odd}).$$

$$\int_x \frac{1}{(\sin x)^n} = -\cos x.\overset{\frac{1}{2}n}{\mathrm{S}}_m \frac{\underline{|n-2}_{\,m-1,-2}}{\underline{|n-1}_{\,m,-2}}(\sin x)^{-(n-2m+1)},\ (n \text{ even}),$$

$$= -\cos x.\overset{\frac{1}{2}(n-1)}{\mathrm{S}}_m \frac{\underline{|n-2}_{\,m-1,-2}}{\underline{|n-1}_{\,m,-2}}(\sin x)^{-(n-2m+1)} + \frac{\underline{|1}_{\,\frac{1}{2}(n-1),2}}{\underline{|2}_{\,\frac{1}{2}(n-1),2}}\log_\epsilon \tan\tfrac{1}{2}x,\ (n \text{ odd}).$$

$$\int_x \frac{1}{(\cos x)^n} = \sin x.\overset{\frac{1}{2}n}{\mathrm{S}}_m \frac{\underline{|n-2}_{\,m-1,-2}}{\underline{|n-1}_{\,m,-2}}\cos x^{-(n-2m+1)},\ (n \text{ even}),$$

$$= \sin x.\overset{\frac{1}{2}(n-1)}{\mathrm{S}}_m \frac{\underline{|n-2}_{\,m-1,-2}}{\underline{|n-1}_{\,m,-2}}\cos x^{-(n-2m+1)} + \frac{\underline{|1}_{\,\frac{1}{2}(n-1),2}}{\underline{|2}_{\,\frac{1}{2}(n-1),2}}\log_\epsilon \cot\left(45^\circ - \tfrac{1}{2}x\right),\ (n \text{ odd}).$$

$$\int_x \epsilon^{ax}(\cos x)^n = \frac{\epsilon^{ax}(\cos x)^{n-1}(a\cos x + n\sin x)}{a^2+n^2} + \frac{n(n-1)}{a^2+n^2}\int_x \epsilon^{ax}(\cos x)^{n-2}.$$

$$\int_x \epsilon^{ax}.\sin bx = \frac{\epsilon^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}.$$

$$\int_x \epsilon^{ax}.\cos bx = \frac{\epsilon^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}.$$

From these formulæ $\int_x \epsilon^{ax}.(\sin bx)^n.(\cos cx)^n$ may be obtained, if $(\sin bx)^n$ and $(\cos cx)^n$ be expanded in series of multiple sines and cosines.

(22.) $\displaystyle\int_x \frac{1}{a+b\cos x}$

$$= \frac{1}{(b^2-a^2)^{\frac12}}\log_\epsilon \frac{\overline{(b+a)(1+\cos x)}\big|^{\frac12} + \overline{(b-a)(1-\cos x)}\big|^{\frac12}}{\overline{(b+a)(1+\cos x)}\big|^{\frac12} - \overline{(b-a)(1-\cos x)}\big|^{\frac12}} + \textit{const.},$$

$$= \frac{1}{(b^2-a^2)^{\frac12}}\log_\epsilon \frac{b+a\cos x+\sin x(b^2-a^2)^{\frac12}}{a+b\cos x} + \textit{const.} \text{ if } a<b:$$

$$= \frac{1}{a}.\tan\tfrac12 x + \textit{const.}, \text{ if } a=b:$$

$$= \frac{2}{(a^2-b^2)^{\frac12}}\tan^{-1}\overline{\frac{(a-b)(1-\cos x)}{(a+b)(1+\cos x)}}\Bigg|^{\frac12} + \textit{const.},$$

$$= \frac{1}{(a^2-b^2)^{\frac12}}\cos^{-1}\frac{b+a\cos x}{a+b\cos x} + \textit{const.} \text{ if } a>b.$$

$$\int_x \frac{\sin x}{a+b\cos x} = -\frac{1}{b}\log_\epsilon(a+b\cos x) + \textit{const.}$$

$$\int_x \frac{\cos x}{a+b\cos x} = \frac{x}{b} - \frac{a}{b}\int_x \frac{1}{a+b\cos x}.$$

$$\int_x \frac{a_1+b_1\cos x}{(a+b\cos x)^n} = \frac{(ab_1-a_1b)\sin x}{(n-1)(a^2-b^2)(a+b\cos x)^{n-1}}$$

$$+\frac{1}{(n-1)(a^2-b^2)}\int_x \frac{(n-1)(aa_1-bb_1)+(n-2)(ab_1-a_1b)\cos x}{(a+b\cos x)^{n-1}}.$$

$$\int_x (1+e.\cos x)^m = Ax + A_1 \sin x + \tfrac{1}{2}A_2 \sin 2x + \tfrac{1}{3}A_3 \sin 3x + \&c.$$

$$A = 1 + \frac{m(m-1)}{1.\ 2} . \frac{1}{2} e^2 + \frac{m\ldots(m-3)}{1\ldots\ 4} . \frac{1.3}{2.4} e^4 + \&c.$$

$$A_1 = 2e \left\{ \frac{m}{1} . \frac{1}{2} + \frac{m(m-1)(m-2)}{1.\ 2\ .\ 3} . \frac{1.3}{2.4} e^2 \right.$$
$$\left. + \frac{m\ldots(m-4)}{1\ldots\ 5} . \frac{1.3.5}{2.4.6} e^4 + \&c. \right\}$$

$$A_2 = \frac{2meA - 2A_1}{(m+2)e}, \qquad A_3 = \frac{(m-1)eA_1 - 2^2 A_2}{(m+3)e},$$

$$A_4 = \frac{(m-2)eA_2 - 2^3 A_3}{(m+4)e}, \quad \&c. = \&c.$$

If $\int_x (1+e.\cos x)^{-m} = Ax + A_1 \sin x + \frac{1}{2} A_2 \sin 2x + \&c.$
and $\int_x (1-e.\cos x)^{-m-1} = Bx + B_1 \sin x + \frac{1}{2} B_2 \sin 2x + \&c.,$
then
$$B = \frac{2mA - (m-1)eA_1}{2m(1-e^2)} = A + \frac{e}{m} d_e A,$$
$$B_1 = \frac{2A - B}{e} = A_1 + \frac{e}{m} d_e A_1,$$
$$B_2 = \frac{2(A_1 - B_1) - 2eB}{e} = A_2 + \frac{e}{m} d_e A_2,$$
$$\&c. = \&c. = \&c.$$

$$\int_x \log_\epsilon (1 + e.\cos x) = -x \log_\epsilon \frac{2k}{e}$$
$$+ 2\{k.\cos x - \tfrac{1}{2}k^2.\cos 2x + \tfrac{1}{3}k^3.\cos 3x - \&c.\}$$

where $k = \frac{1}{e}\{1 - (1-e^2)^{\frac{1}{2}}\}$. (*L. C. D.* 449—66.)

APPROXIMATE VALUES OF INTEGRALS.

(23.) Let $\int_{x=a} u$ represent the value of $\int_x u$, when $x = a$; then

$$\int_{x=a+b} u - \int_{x=a} u = (u)_{x=a} \frac{b}{1} + d_{x=a} u . \frac{b^2}{1.2} + d^2_{x=a} u . \frac{b^3}{1.2.3} + \&c.$$

This series does not converge with sufficient rapidity, unless b is very small: the following more convergent series may however be obtained, by dividing the difference between the limits into n equal intervals;

$$\begin{aligned}
\int_{x=a+nb} u - \int_{x=a} u = \ & \frac{b}{1}\{(u)_{x=a} + (u)_{x=a+b} + \&c. + (u)_{x=a+(n-1)b}\} \\
& + \frac{b^2}{1.2}\{d_{x=a}u + d_{x=a+b}u + \&c. + d_{x=a+(n-1)b}u\} \\
& + \frac{b^3}{1.2.3}\{d^2_{x=a}u + d^2_{x=a+b}u + \&c. + d^2_{x=a+(n-1)b}u\} \\
& + \&c. \\
= \ & \frac{b}{1}\{(u)_{x=a+b} + (u)_{x=a+2b} + \&c. + (u)_{x=a+nb}\} \\
& - \frac{b^2}{1.2}\{d_{x=a+b}u + d_{x=a+2b}u + \&c. + d_{x=a+nb}u\} \\
& + \frac{b^3}{1.2.3}\{d^2_{x=a+b}u + d^2_{x=a+2b}u + \&c. + d^2_{x=a+nb}u\} \\
& - \&c. \quad \text{[α]} \\
= \ & \frac{b}{1}\{(u)_{x=a+b} + \&c. + (u)_{x=a+(n-1)b} + \tfrac{1}{2}.\overline{(u)_{x=a} + (u)_{x=a+nb}}\} \\
& + \frac{b^2}{1.2}.\tfrac{1}{2}\{d_{x=a}u - d_{x=a+nb}u\} \\
& + \&c.
\end{aligned}$$

The values of $\int_x u$ between the limits $x=a$, and $x=a+nb$ determined from the first series are always less, and from the second, alternately greater and less than the true value. The third series, which is half the sum of the two first, is more convenient for calculation.

This method is inapplicable, if any value of x between the assigned limits renders any of the differential coefficients infinite.

(*L. C. D.* 467—73; *Tr. L.* 209—13.)

[α] $$\begin{aligned}
\int_{x=a+nb} u - \int_{x=a} u &= \overset{\infty}{\mathrm{S}}_r \frac{b^r}{\lfloor\underline{r}} . \overset{n}{\mathrm{S}}_s d^{r-1}_{x=a+(s-1)b} u, \\
&= \overset{\infty}{\mathrm{S}}_r (-1)^{r-1} \frac{b^r}{\lfloor\underline{r}} . \overset{n}{\mathrm{S}}_s d^{r-1}_{x=a+sb} u.
\end{aligned}$$

SUCCESSIVE INTEGRATION.

(24.) $\int_x^2 u = x\int_x u - \int_x ux.$

$$\int_x^3 u = \frac{1}{1.2}\{x^2\int_x u - 2x\int_x ux + \int_x ux^2\}.$$

$$\int_x^4 u = \frac{1}{1.2.3}\{x^3\int_x u - 3x^2\int_x ux + 3x\int_x ux^2 - \int_x ux^3\}$$

&c. = &c.

$$\int_x^n u = \frac{1}{1.2\ldots(n-1)}\Big\{x^{n-1}\int_x u - \frac{n-1}{1}x^{n-2}\int_x ux$$
$$+ \frac{(n-1)(n-2)}{1\ .\ 2}x^{n-3}\int_x ux^2 - \&c. + (-1)^{n-1}\int_x ux^{n-1}\Big\} \quad {}^{\alpha}$$

(25.) *Series independent of the integral sign.*

$$\int_x^2 u = \frac{x^2}{1}\left\{\frac{1}{2}u - \frac{x}{1.3}.d_x u + \frac{x^2}{1.2.4}.d_x^2 u - \&c.\right\} + c_1 x + c_2.$$

$$\int_x^3 u = \frac{x^3}{1.2}\left\{\frac{1}{3}u - \frac{x}{1.4}d_x u + \frac{x^2}{1.2.5}d_x^2 u - \&c.\right\} + c_1 x^2 + c_2 x + c_3.$$

&c. = &c.

$$\int_x^n u = \frac{x^n}{1.2\ldots(n-1)}\left\{\frac{1}{n}u - \frac{x}{1(n+1)}d_x u + \frac{x^2}{1.2(n+2)}d_x^2 u - \&c.\right\}$$
$$+ c_1 x^{n-1} + c_2 x^{n-2} + \&c. + c_n; \quad {}^{\beta}$$

c_1, c_2 &c. being arbitrary functions. (*L. C. D.* 483—5.)

INTEGRATION OF FUNCTIONS OF SEVERAL VARIABLES.

(26.) If u is a function of x and y, then

$$u = \int_x d_x u + \phi(y),$$

an arbitrary function of y, which would have disappeared by differentiation, being added instead of an arbitrary constant.

$${}^{\alpha}\int_x^n u = \frac{1}{\underline{|n-1}}.\overset{n}{\mathrm{S}}_m(-1)^{m-1}\frac{\overset{\underline{|n-1}}{\scriptstyle m-1}}{\underline{|m-1}}x^{n-m}\int_x ux^{m-1}.$$

$${}^{\beta}\int_x^n u = \frac{x^n}{\underline{|n-1}}.\overset{\infty}{\mathrm{S}}_m(-1)^{m-1}\frac{x^{m-1}}{n+m-1}d_x^{m-1}:u + \overset{n}{\mathrm{S}}_r c_r x^{n-r}.$$

If $d_x d_y u = v$, then $d_x u = \int_y v + f(x)$; and

$$u = \int_x \int_y v + \phi_1(x) + \phi_2(y).$$

Similarly if $d_x d_y d_z u = v_1$ then

$$u = \int_x \int_y \int_z v + \phi_1(x) + \phi_2(y) + \phi_3(z);$$

and so on, whatever may be the number of variables.

(*L. D. C.* 279—81.)

DIFFERENTIAL EQUATIONS.

(27.) If any equation $\phi(x,y) = 0$ contains n independent constants, then

[1] n differential equations of the first *order* may be obtained, from each of which one of the n constants has disappeared:

[2] any one of these n equations will be of the same degree with respect to the differential coefficient, as the highest power of the constant which has disappeared:

[3] there will be $\dfrac{n(n-1)(n-2)\ldots(n-m+1)}{1 \, . \, 2 \, . \, 3 \ldots m}$ differential equations of the m^{th} order, each involving $n-m$ of the n constants:

[4] there will be but one differential equation of the n^{th} order, which contains no constant.

From any differential equation of the m^{th} order m equations of the $(m-1)^{\text{th}}$ order may be obtained, in each of which one constant has been introduced. (*L. D. C.* 287—304.)

(28.) *The criterion of integrability.* If the equation

$$M + N d_x y = 0$$

can be obtained by differentiation alone from the equation $u = 0$, then

$$d_y M = d_x N,$$

and

$$u = \int_y N + \int_x (M - d_x \int_y N).$$

$\int_x d_y M = d_y \int_x M$; by means of which either of the coefficients M, N, may be obtained from the other.

If the equations $M+Nd_xy=0$, $M+Pd_xz=0$, $N+Pd_yz=0$, (1) may either of them be derived from the same function u, then

$$d_yM=d_xN, \qquad d_zM=d_xP, \qquad d_zN=d_yP;$$

and u_1, u_2, u_3 being the integrals obtained from the equations (1) respectively,

$$\begin{aligned} u &= u_1+\int_z(P-d_zu_1), \\ &= u_2+\int_y(N-d_yu_2), \\ &= u_3+\int_x(M-d_xu_3). \quad (L.D.C.\ 282\text{—}6.) \end{aligned}$$

(29.) *Separation of the variables.* All equations of the form

$$X+Yd_xy=0,$$

in which X and Y are functions of x and y respectively, are immediately integrable; the integral is

$$\int_x X+\int_y Y=0.$$

The equations $Y+Xd_xy=0$, and $X_1Y+XY_1d_xy=0$ may be reduced to the above form by dividing them by XY.

(30.) *Homogeneous equations.* Any homogeneous equation between two variables, each term of which is of m dimensions may be rendered integrable by dividing every term by x^m, and assuming $y=zx$, then

$$d_xy=z+xd_xz,$$

and the given equation may be reduced to

$$Z_1+Z_2(z+xd_xz)=0,$$

in which the variables may readily be separated.

The equation $a_1+b_1x+c_1y+(a_2+b_2x+c_2y)d_xy=0$ may be rendered homogeneous by assuming

$$a_1+b_1x+c_1y=u, \qquad a_2+b_2x+c_2=v,$$

whence $$d_xy=-\frac{b_1.d_uv-b_2}{c_1.d_uv-c_2}.$$

(31.) *The linear equation,* $d_xy+X_1y+X_2=0$.

This equation may be rendered integrable by multiplying every term by $e^{\int_x X_1}$: the required integral is

$$y=e^{-\int_x X_1}\{\int_x X_2e^{\int_x X_1}+const.\}$$

(32.) ***Riccati's equation***; $d_x y + ay^2 + bx^m = 0$.

If $m = 0$, the variables are immediately separable: otherwise assume $y = \frac{z}{x^2} + \frac{1}{ax}$, then

$$d_x y = \frac{d_x z}{x^2} - \frac{2z}{x^3} - \frac{1}{ax^2}:$$

substituting these values for y and $d_x y$ respectively, we obtain

$$x^2 d_x z + az^2 + bx^{m+4} = 0.$$

This equation is homogeneous, if $m = -2$; and if $m = -4$, the variables are immediately separable: otherwise assume

$$z = \frac{1}{y_1}, \qquad x^{m+3} = x_1,$$

and by substituting these values, the equation will be reduced to its original form, and the same process may be repeated.

The given equation is integrable if $m = \frac{-4n}{2n \pm 1}$, the value of n being 0, or any positive integer.

THE INTRODUCTION OF A FACTOR WHICH RENDERS A DIFFERENTIAL EQUATION INTEGRABLE.

(33.) Let $M + N d_x y = 0$ be an equation which does not fulfil the condition

$$d_y M = d_x N,$$

and let $V = c$ be the primitive equation, and

$$\frac{d_x V}{u} = M, \qquad \frac{d_y V}{u} = N,$$

then the given equation may be rendered integrable by introducing the factor $u . \phi(u)$: in the simplest case $\phi(u) = 1$.

To determine u we have

$$u\{d_y M - d_x N\} = N d_x u - M d_y u,$$

this however usually presents greater difficulties than the given equation.

If u is a function of x only, then $d_y u = 0$, and

$$\frac{d_x u}{u} = \frac{1}{N}(d_y M - d_x N) = X$$

hence $$u = \epsilon^{\int_x X}.$$

If u is a function of y only, it may be determined in a similar manner.

If M and N are homogeneous functions, then

$$u = \frac{1}{Mx + Ny}.$$

(34.) The required factor may sometimes be conveniently found by transforming the equation

$$M + N d_x y = 0 \text{ into } K + L d_t u = 0,$$

in which the relations between x, y, t, and u, are known, and finding a factor V such that $\frac{K}{V}$ may be a function of t, and $\frac{L}{V}$ of u: then if the values of t and u, in terms of x and y, be substituted in V, the quantity obtained will render the given equation integrable.

The equation $P + Q d_t u = 0$ may be rendered integrable by the factor $\frac{1}{PT - QU}$ if

$$P = \frac{V}{T}\phi_1\left(\int_t \frac{1}{T} - \int_u \frac{1}{U}\right),$$

$$Q = \frac{V}{U}\phi_2\left(\int_t \frac{1}{T} - \int_u \frac{1}{U}\right);$$

V being any function of t and u, and T and U, of t and u, respectively.

The equation $X_1 y + (y + X_2) d_x y = 0$ may be rendered integrable by the factor $\frac{1}{y^3 + Xy^2 + \{X - a + b(2a - X)^2\} ay}$.

The equation $X_1 y + (X_2 y + X_3) d_x y = 0$ may be rendered integrable by the factor $\frac{y^m}{(1 + Xy)^n}$ if

$$X_1 = bX^m . d_x X + (n-1)aX^{n-2} . d_x X,$$
$$X_2 = (m-n+2)aX^n,$$
$$X_3 = bX^{m+1} + (m+1)aX^{n-1}.$$

(*L. D. C.* 316—23; *L. C. D.* 567—80.)

SINGULAR SOLUTIONS.

(35.) Let $f(x,y,c) = 0$, c being an arbitrary constant, and let $\phi(x,y,y',c) = 0$, in which y' represents $d_x y$, be derived from the former by differentiation; from these equations, another of the form $\psi(x,y,y') = 0$ may be obtained by the elimination of c.

If u is such a function of x and y, that

$$\phi(x,y,y',u) = 0,$$

may be derived by differentiation from

$$f(x,y,u) = 0,$$

then $\psi(x,y,y') = 0$ will result from the elimination of u, and $f(x,y,u) = 0$ may be considered as the primitive equation.

Such values of $f(x,y,u) = 0$ as are not included in the general form $f(x,y,c) = 0$ are *singular solutions.*

The equation $d_u f(x,y,u) = 0$ must be satisfied by the values of u which give singular solutions.

All singular solutions will be found amongst the equations obtained by substituting in $f(x,y,c) = 0$ those variable values of c which satisfy the equation

$$d_c f(x,y,c) = 0.$$

The values of c which give singular solutions satisfy the equations $\quad d_x c = \infty, \quad d_y c = \infty;$

they also render $d_x^2 y = \frac{0}{0}$.

If $f(x,y,c) = 0$, the general solution of a differential equation, represents a class of curves, each of which is determined by assigning a particular value to the arbitrary constant c, then the singular solution, which is independent of c, represents the bounding or circumscribing curve.

If a given differential equation $f(x,y,y')=0$ can be solved for y', then a singular solution may be obtained by substituting for y' a value which satisfies the equations

$$d_x y' = \infty\,, \qquad d_y y' = \infty\,.$$

(*L. D. C.* 324—41.)

For singular solutions of differential equations of higher orders, see *L. C. D.* 638—58.

(36.) *Equations of more than one dimension in y'.*

If a constant of n dimensions be eliminated between any equation $f(x,y)=0$ and its derived equation $f'(x,y)=0$, an equation may be obtained of the form

$$y'^n + a_{n-1}y'^{n-1} + \&c. + a_1 y' + a = 0:$$

let the roots of this be p_1, p_2, &c. and let the equations

$$y' - p_1 = 0, \qquad y' - p_2 = 0, \ \&c.$$

be integrated separately; then the primitive equation will be the product of the integrals thus obtained.

If we have an equation $f(x,y,y')=0$, homogeneous with regard to x and y, then by assuming $y = ux$, we obtain

$$x = \epsilon^{\int_u \frac{1}{y'-u}}, \qquad y = u\epsilon^{\int_u \frac{1}{y'-u}},$$

whence we may frequently obtain an integrable equation between y' and u.

(37.) *Clairault's formula:* $y = y'x + f(y')$.

By differentiating the given equation we obtain

$$\{x + d_{y'} f(y')\}\, d_x^2 y = 0,$$

whence $d_x^2 y = 0$, and $\therefore$ $y' = c$, by substituting which we have

$$y = cx + f(c).$$

This is the general solution: a singular solution may be obtained by eliminating y' between the original equation, and

$$x + d_{y'} f(y') = 0.$$

(*L. D. C.* 342—7; *L. C. D.* 582—9.)

(38.) Let the equation be $f(d_x^2y,x)=0$.

If this equation can be solved for d_x^2y, another may be obtained of the form $d_x^2y=X$;

whence $d_xy=\int_x X+c_1$,

and $y=\int_x^2 X+c_1x+c_2$.

Let the equation be $f(d_x^2y,y)=0$.

If this can be solved for d_x^2y, we obtain

$$d_x^2y=Y;$$

whence, since $d_x^2y=y'd_yy'$, $(d_xy)^2=2\int_y Y+c$,

and $x=\int_y(2\int_y Y+c)^{-\frac{1}{2}}+c_1$.

Let the equation be $f(d_x^2y,d_xy)=0$:

this being solved for d_x^2y gives

$$d_x^2y=\phi(y'), \qquad \text{or } d_xy'=\phi(y'),$$

whence $x=\int_{y'}\frac{1}{\phi(y')}$, and $y=\int_{y'}\frac{y'}{\phi(y')}$;

the primitive equation between x and y may be obtained by eliminating y' between these two equations, each of which involves one arbitrary constant.

Let the equation be $f(d_x^2y,d_xy,x)=0$.

This is a differential equation of the first order with respect to x and d_xy, the integral of which is of the form

$$\phi(d_xy,x,c_1)=0, \qquad (1)$$

and from this may be obtained the primitive equation

$$\psi(x,y,c_1,c_2)=0.$$

If (1) can be solved for d_xy we obtain

$$d_xy=X, \qquad \text{and hence } y=\int_x X.$$

If (1) can be solved for x only, then we have

$$x=f(y'),$$

and by integrating by parts,

$$y=y'x-\int_{y'}f(y'):$$

from these two equations the primitive equation may be obtained by elimination.

Let the equation be $f(d_x^2y, d_xy, y) = 0$.

If this equation can be solved for d_x^2y, we have

$$d_x^2y = y' d_y y' = \phi(y', y),$$

which is an equation of the first order, from which we may obtain $$\psi(y', y, c) = 0.$$

If this can solved for y', we have $y' = Y$, which is immediately integrable: if it can be solved for y only, then we obtain

$$y = \phi(y'),$$

hence $y' = d_x\phi(y')$, and $x = \int_{y'} \dfrac{d_{y'}\phi(y')}{y'}$:

from these the primitive equation may be obtained by eliminating y'. (*L. D. C.* 353—60; *Tr. L.* 273—7.)

(39.) Let the equation be $d_x^2y + n^2y + X = 0$;
X being a function of x and constants only.

By multiplying by $\cos nx$, and integrating by parts, we obtain $\cos nx . d_xy + ny \sin nx + \int_x \cos nx . X = 0$:
this being divided by $(\cos nx)^2$ and integrated, the result is

$$y = -\cos nx \int_x \frac{1}{(\cos nx)^2} \int_x \cos nx . X.$$

If $X = 0$, then $y = \dfrac{c_1}{n} \sin nx + c_2 \cos nx$.

If $X = -a$, then $y = \dfrac{a}{n^2} + \dfrac{c_1}{n} \sin nx + c_2 \cos nx$.

If $X = b \cos(mx + B)$, then

$$y = \frac{b}{m^2 - n^2} \cos(mx + B) + \frac{c_1}{n} \sin nx + c_2 \cos nx.$$

If $X = b \cos(nx + B)$, then

$$y = -\frac{b}{2} . \frac{x}{n} \sin(nx + B) + \frac{c_1}{n} \sin nx + c_2 \cos nx.$$

(*Airy*, *Math. Tracts.*)

(40.) Let $d_x^2 y + X_1 d_x y + X_2 y = 0$:
assume $y = \epsilon^{\int_x u}$, then by substitution

$$d_x u + u^2 + X_1 u + X_2 = 0;$$

an equation of the first order, which is not however generally integrable, unless X_1 and X_2 are constants.

$$\text{Let } d_x^2 y + X_1 d_x y + X_2 y + X_3 = 0.$$

assume $y = tu$, and substitute for $d_x^2 y$, $d_x y$, and y, their values in terms of t and u; then u being determined by assuming the condition

$$d_x^2 u + X_1 d_x u + X_2 = 0,$$

the transformed equation becomes

$$d_x^2 t + d_x t\left(X_1 + \frac{2}{u} \cdot d_x u\right) + \frac{X_3}{u} = 0.$$

and this last equation, by assuming $v = \epsilon^{\int_x X_1}$ becomes

$$d_x^2 t + d_x t\left(\frac{1}{v} d_x v + \frac{2}{u} d_x u\right) + \frac{X_3}{u} = 0,$$

whence $$d_x t \,.\, v u^2 + \int_x X_3 v u = 0,$$
from which may be obtained the primitive equation

$$y = u \int_x \frac{1}{vu^2} \int_x v u X_3.$$

(41.) If the given equation is homogeneous with respect to x and y, assume

$$y = ux, \qquad d_x^2 y = \frac{v}{x}, \qquad (1)$$

then the equation becomes $f(y', v, u) = 0$:
from this last equation solved for v, and the equations (1), we have $$d_u y' - \frac{1}{y' - u} f_1(y', u) = 0;$$

hence by integration $y' = f_2(u, c_1)$, and finally

$$x = c_2 \epsilon^{\varphi\left(\frac{y}{x}, c_1\right)}.$$

(*L. D. C.* 361—4; *Tr. L.* 278—83.)

(42.) Differential equations which do not contain either variable, and are of the form

$$f(d_x^n y,\ d_x^{n-1}y)=0,$$

may be reduced to $\quad f(d_xu,u)=0,$

by assuming $d_x^{n-1}y=u$, whence $d_x^ny=d_xu$.

Equations of the form $f(d_x^ny,\ d_x^{n-2}y)=0$ may in a similar manner by assuming $d_x^{n-2}y=u$ be reduced to

$$f(d_x^2u,u)=0.$$

By the same assumption the equation

$$f(d_x^ny,\ d_x^{n-1}y,\ d_x^{n-2}y)=0$$

may be reduced to the form

$$f(d_x^2u,\ d_xu,\ u)=0:$$

this and the two former equations in u may be integrated by preceding methods.

(43.) *Solution of the equation*

$$d_x^ny+A_1d_x^{n-1}y+A_2d_x^{n-2}y+\&c.+A_{n-1}d_xy+A_ny=0. \qquad (1)$$

Assume $y=\epsilon^u$, then

$$d_xy=\epsilon^ud_xu;\quad d_x^2y=\epsilon^u\{d_x^2u+(d_xu)^2\};\ \&c.$$

by substituting these values of u and its differential coefficients, and dividing by ϵ^u, we obtain

$$d_x^nu+B_1d_x^{n-1}u+\&c.+B_{n-1}d_xu+A_n=0; \qquad (2)$$

where $\quad B_{n-1}=(d_xu)^{n-1}+A_1(d_xu)^{n-2}+\&c.+A_{n-1}$.

This equation is seldom integrable; if however A_1, A_2, &c. the coefficients of (1) are constant, (2) will be satisfied by a constant value of d_xu; in which case since $d_x^2u=0$, $d_x^3u=0$, &c.

$$(d_xu)^n+A_1(d_xu)^{n-1}+A_2(d_xu)^{n-2}+\ldots A_{n-1}d_xu+A_n=0.$$

let the roots of this equation be $\alpha_1\ \alpha_2$ &c., then

from $d_xu=\alpha_1$ we obtain $y=c_1\epsilon^{\alpha_1x}$,
.... $d_xu=\alpha_2$ .... $y=c_2\epsilon^{\alpha_2x}$,
&c. .... &c.

and $\qquad y=C_1\epsilon^{\alpha_1x}+C_2\epsilon^{\alpha_2x}+\&c.+C_n\epsilon^{\alpha_nx}$

is the complete integral required.

If two roots, as a_1, a_2, $\text{⫩}\; a \pm b\sqrt{-1}$, then the corresponding part of the value of y

$$\text{⫩}\; k_1 \epsilon^{ax} \sin(bx + k_2).$$

If $a_1 = a_2 = \&\text{c.} = a_p$, the corresponding part of the value of y

$$\text{⫩}\; \epsilon^{a_1 x}\{K_1 + K_2 x + \ldots + K_p x^{p-1}\}.$$

If there are two equal impossible roots, the corresponding part of the value of y

$$\text{⫩}\; \epsilon^{a_1 x}\{(F_1 + F_2 x)\cos bx + (G_1 + G_2 x)\sin bx\}.$$

(44.) *Solution of the equation*

$$d_x^n y + A_1 d_x^{n-1} y + \ldots + A_{n-1} d_x y + A_n y + X = 0. \qquad (1)$$

[1] *Euler's solution.* Let ϵ^{-kx} be the factor which will render this equation integrable, and let

$$\textstyle\int_x - X\epsilon^{-kx} = X_1 + K,$$

and $\int_x \epsilon^{-kx}\{d_x^n y + A_1 d_x^{n-1} y + \ldots + A_{n-1} d_x y + A_n y\}$

$$= \epsilon^{-kx}\{d_x^{n-1} y + B_1 d_x^{n-2} y + \ldots + B_{n-1} y\};$$

by differentiating which, and equating coefficients, B_1, &c. may be determined in terms of A_1, &c. and k will have n values determined by the equation

$$k^n + A_1 k^{n-1} + \ldots + A_{n-1} k + A_n = 0,$$

the equation (1) will therefore have n immediate integrals of the form

$$d_x^{n-1} y + B_1 d_x^{n-2} y + \ldots + B_{n-1} y = \epsilon^{kx}\{X_1 + K\},$$

the value of X_1 depending on that of k: from these n equations the $n-1$ differential coefficients may be eliminated, and the result will be the primitive equation.

[2] *Lagrange's solution.* Let the equation

$$y = c_1 \epsilon^{a_1 x} + c_2 \epsilon^{a_2 x} + \&\text{c.} = 0 \qquad (2)$$

be the integral of

$$d_x^n y + A_1 d_x^{n-1} y + \ldots + A_{n-1} d_x y + A_n y = 0;$$

it is required to determine into what functions of x, c_1, c_2, &c. must be changed, that (2) may be the complete integral of (1): in order to render the investigation more intelligible, it may be applied to an equation of the third order,

$$d_x^3y + A_1d_x^2y + A_2d_xy + A_3y + X = 0. \qquad (3)$$

Let $$y = X_1y_1 + X_2y_2 + X_3y_3$$

be the required integral, in which y_1, y_2, y_3, represent ϵ^{α_1x}, ϵ^{α_2x}, ϵ^{α_3x} respectively. By differentiating this equation we obtain

$$d_xy = X_1.d_xy_1 + X_2.d_xy_2 + X_3.d_xy_3 + y_1.d_xX_1 + y_2.d_xX_2 + y_3.d_xX_3;$$

assume $$y_1.d_xX_1 + y_2.d_xX_2 + y_3.d_xX_3 = 0, \qquad (4)$$

then by differentiating again, we obtain

$$d_x^2y = X_1.d_x^2y_1 + X_2.d_x^2y_2 + X_3.d_x^2y_3$$
$$+ d_xy_1.d_xX_1 + d_xy_2.d_xX_2 + d_xy_3.d_xX_3;$$

assume $$d_xy_1.d_xX_1 + d_xy_2.d_xX_2 + d_xy_3.d_xX_3 = 0, \qquad (5)$$

and by repeating the differentiation,

$$d_x^3y = X_1.d_x^3y_1 + X_2.d_x^3y_2 + X_3.d_x^3y_3$$
$$+ d_x^2y_1.d_xX_1 + d_xy_2.d_xX_2 + d_x^2y_3.d_xX_3;$$

substituting these values of d_xy, d_x^2y, d_x^3y in (1), we have

$$d_x^2y_1.d_xX_1 + d_x^2y_2.d_xX_2 + d_x^2y_3.d_xX_3 + X = 0, \qquad (6)$$

since either of the three values of y will satisfy the equation

$$d_x^3y + A_1d_x^2y + A_2d_xy + A_3y = 0:$$

from (4), (5), (6), X_1, X_2, and X_3 may be determined by elimination, and thus the primitive equation may be obtained.

If y_3 is unknown then (5) is inadmissible; X_1, X_2, X_3 may however be determined by the solution of a differential equation of the first order: if y_2, and y_3 are both unknown, (4) is likewise inadmissible; the values of X_1, X_2, X_3 may however be found by means of a differential equation of the second order.

The same method of investigation is applicable to the equation of the n^{th} order: and if only p particular values of y can be found, the integration of a differential equation of the $(n-p)^{\text{th}}$ order must be effected in order to determine the quantities X_1, X_2, &c.

(45.) The complete integral of the equation (1) is

$$y=\frac{\epsilon^{a_1 x}\left(c_1-\int_x X\epsilon^{-a_1 x}\right)}{(a_1-a_2)(a_1-a_3)\ldots(a_1-a_n)}+\frac{\epsilon^{a_2 x}\left(c_2-\int_x X\epsilon^{-a_2 x}\right)}{(a_2-a_1)(a_2-a_3)\ldots(a_2-a_n)}$$

$$+\ \&\text{c.}+\frac{\epsilon^{a_n x}\left(c_n-\int_x X\epsilon^{-a_n x}\right)}{(a_n-a_1)(a_n-a_2)\ldots(a_n-a_{n-1})}.$$

If two of the quantities a_1, a_2, &c. $= a \pm b\sqrt{-1}$, the two corresponding terms in the values of y become

$$\frac{2\epsilon^{ax}(A\cos bx+B\sin bx)\left(c_1-\cos bx\int_x X\epsilon^{-ax}\right)}{A^2+B^2}$$

$$+\frac{2\epsilon^{ax}(A\sin bx-B\cos bx)\left(c_2-\sin bx\int_x X\epsilon^{-ax}\right)}{A^2+B^2},$$

$A+B\sqrt{-1}$, and $A-B\sqrt{-1}$ being the denominators of the fractions.

If two of the above quantities are equal, the corresponding terms become

$$\frac{\epsilon^{a_1 x}\left(\int_x Xx\epsilon^{-a_1 x}-x\int_x X\epsilon^{-a_1 x}\right)}{E_2}-d_{a_1}\frac{1}{E_2}\epsilon^{a_1 x}\int_x X\epsilon^{-a_1 x};$$

where $E_2=(a_1-a_3)(a_1-a_4)\ldots(a_1-a_n)$.

If $a_1=a_2=a_3$, then the corresponding value of y becomes

$$-\frac{\epsilon^{a_1 x}}{2E_3}\left(\int_x Xx^2\epsilon^{-a_1 x}-2x\int_x X.x\epsilon^{-a_1 x}+x^2\int_x X\epsilon^{-a_1 x}\right)$$

$$+\ \epsilon^{a_1 x}\left(\int_x Xx\epsilon^{-a_1 x}-x\int_x X\epsilon^{-a_1 x}\right)d_{a_1}\frac{1}{E_3}$$

$$-\tfrac{1}{2}\epsilon^{a_1 x}\int_x X\epsilon^{-a_1 x}.\,d_{a_1}{}^2\frac{1}{E_3};$$

where $E_3=(a_1-a_4)(a_1-a_5)\ldots(a_1-a_n)$.

(*L. D. C.* 370—8; *L. C. D.* 603—14.)

SIMULTANEOUS EQUATIONS.

(46.) Let there be three equations,

$$\begin{aligned} a_1 u + a_2 x + a_3 y + a_4 d_t u + a_5 d_t x + a_6 d_t y &= 0, \\ b_1 u + b_2 x + b_3 y + b_4 d_t u + b_5 d_t x + b_6 d_t y &= 0, \qquad (1) \\ c_1 u + c_2 x + c_3 y + c_4 d_t u + c_5 d_t x + c_6 d_t y &= 0, \end{aligned}$$

in which the coefficients a_1, $a_2, \ldots b_1$, b_2, &c. are constants.

Assume $$x = mu, \qquad y = nu,$$

then in order that the equations (1) may subsist,

$$\frac{a_1 + a_2 m + a_3 n}{a_4 + a_5 m + a_6 n} = \frac{b_1 + b_2 m + b_3 n}{b_4 + b_5 m + b_6 n} = \frac{c_1 + c_2 m + c_3 n}{c_4 + c_5 m + c_6 n}.$$

Let $$\frac{a_1 + a_2 m + a_3 n}{a_4 + a_5 m + a_6 n} = k,$$

then an equation of three dimensions may be obtained for determining k; let the roots of this equation be k_1, k_2, k_3, and let m_1, m_2, m_3, and n_1, n_2, n_3, be the corresponding values of m, and n, and C_1, C_2, C_3 arbitrary constants; then

$$\begin{aligned} u &= C_1 \epsilon^{-k_1 t} + C_2 \epsilon^{-k_2 t} + C_3 \epsilon^{-k_3 t}, \\ x &= C_1 m_1 \epsilon^{-k_1 t} + C_2 m_2 \epsilon^{-k_2 t} + C_3 m_3 \epsilon^{-k_3 t}, \\ y &= C_1 n_1 \epsilon^{-k_1 t} + C_2 n_2 \epsilon^{-k_2 t} + C_3 n_3 \epsilon^{-k_3 t}. \end{aligned}$$

The equations

$$\begin{aligned} a_1 u + a_2 x + a_3 y + (a_4 + a_5 t)(a_6 d_t u + a_7 d_t x + a_8 d_t y) &= 0, \\ b_1 u + b_2 x + b_3 y + (b_4 + b_5 t)(b_6 d_t u + b_7 d_t x + b_8 d_t y) &= 0, \\ c_1 u + c_2 x + c_3 y + (c_4 + c_5 t)(c_6 d_t u + c_7 d_t x + c_8 d_t y) &= 0, \end{aligned}$$

may be reduced to the same form as (1) by assuming

$$a + bt = bs, \qquad \text{and } d_r s = s.$$

If any of the quantities k_1, k_2, k_3 are equal, or impossible, the changes which take place in the values of u, x, and y, are analogous to those in the preceding article.

(47.) The equations of the second order

$$\left.\begin{aligned} a_1x + a_2y + a_3d_tx + a_4d_ty + a_5d_t^2x + a_6d_t^2y = 0, \\ b_1x + b_2y + b_3d_tx + b_4d_ty + b_5d_t^2x + b_6d_t^2y = 0, \end{aligned}\right\} \quad (1)$$

may be reduced to the equations of the first order

$$\left.\begin{aligned} a_1x + a_2y + a_3p + a_4q + a_5d_tp + a_6d_tq = 0, \\ b_1x + b_2y + b_3p + b_4q + b_5d_tp + b_6d_tq = 0, \end{aligned}\right\} \quad (2)$$

by assuming $\qquad d_tx = p, \qquad d_ty = q. \qquad (3)$

Let $y = lx,\ p = mx,\ q = nx,$

then in order that the equations (2) and (3) may subsist we must have

$$\frac{a_1 + a_2l + a_3m + a_4n}{a_5m + a_6n} = \frac{b_1 + b_2l + b_3m + b_4n}{b_5m + b_6n} = m = \frac{n}{l}.$$

from these equations we may obtain

$$x = C_1x_1 + C_2x_2 + C_3x_3 + C_4x_4,$$
$$y = C_1y_1 + C_2y_2 + C_3y_3 + C_4y_4,$$

x_1, x_2, y_1, y_2, &c. being the particular values of x and y, corresponding to the roots of an equation of four dimensions.

The solution of the equations

$$a_1u + a_2x + a_3y + a_4d_tu + a_5d_tx + a_6d_ty + T_1 = 0,$$
$$b_1u + b_2x + b_3y + b_4d_tu + b_5d_tx + b_6d_ty + T_2 = 0,$$
$$c_1u + c_2x + c_3y + c_4d_tu + c_5d_tx + c_6d_ty + T_3 = 0;$$

$$a_1x + a_2y + a_3d_tx + a_4d_ty + a_5d_t^2x + a_6d_t^2y + T_1 = 0,$$
$$b_1x + b_2y + b_3d_tx + b_4d_ty + b_5d_t^2x + b_6d_t^2y + T_2 = 0;$$

may be determined from the solution of the same equations wanting the last term by a method analogous to that of Lagrange (Art. 44.): the same methods of solution may be applied to a greater number of equations involving a greater number of independent variables.

(*L. C. D.* 615—9.)

(48.) *D'Alembert's Method.*

$$T_1 d_t x + T_2 d_t y + T_3 x + T_4 y + T_5 = 0, \qquad (1)$$

$$t_1 d_t x + t_2 d_t y + t_3 x + t_4 y + t_5 = 0, \qquad (2)$$

are equations of the first degree between three variables in their most general form, in which t is the independent variable, and T_1, $T_2 \ldots t_1$, t_2, &c. functions of t.

Multiplying (2) by θ an arbitrary function of t, we have

$$(T_1 + t_1\theta)\left\{d_t x + \frac{T_2 + t_2\theta}{T_1 + t_1\theta} . d_t y\right\} + (T_3 + t_3\theta)\left\{x + \frac{T_4 + t_4\theta}{T_3 + t_3\theta} y\right\}$$
$$+ T_5 + t_5\theta = 0: \qquad (3)$$

this equation is linear with respect to

$$x + \frac{T_4 + t_4\theta}{T_3 + t_3\theta} y, \qquad (u)$$

if $$d_t\left\{x + \frac{T_4 + t_4\theta}{T_3 + t_3\theta} y\right\} = d_t x + \frac{T_2 + t_2\theta}{T_1 + t_1\theta} . d_t y, \qquad (4)$$

in which case

$$d_t u + \frac{T_3 + t_3\theta}{T_1 + t_1\theta} u + \frac{T_5 + t_5\theta}{T_1 + t_1\theta} = 0.$$

from (4) we obtain

$$\frac{T_4 + t_4\theta}{T_3 + t_3\theta} = \frac{T_2 + t_2\theta}{T_1 + t_1\theta}, \qquad d_t \frac{T_4 + t_4\theta}{T_3 + t_3\theta} = 0;$$

from which equations θ may be eliminated, and the resulting equation between T_1, t_1, &c. will determine the condition on which the integrability of (1), (2) depends.

The equations (1), (2) may be thus integrated: by alternately eliminating $d_t x$ and $d_t y$, we obtain equations of the form

$$d_t x + U_1 x + U_2 y + U_3 = 0, \qquad (a)$$

$$d_t y + V_1 x + V_2 y + V_3 = 0; \qquad (b)$$

putting u for $x + y\theta$ in $(a) + (b) \times \theta$, we have

$$d_t u + u(U_1 + V_1\theta) - y\{d_t\theta + (U_1 + V_1\theta)\theta - (U_2 + V_2\theta)\}$$
$$+ U_3 + V_3\theta = 0.$$

Let θ be determined by the equation

$$d_t\theta + V_1\theta^2 + (U_1 - V_2)\theta - U_2 = 0,$$

and let the value thus found be substituted in

$$d_t u + u(U_1 + V_1\theta) + U_3 + V_3\theta = 0, \qquad (c)$$

which may be integrated by preceding methods.

If U_1, V_1, are constant, then $d_t\theta = 0$, and

$$\theta^2 + \frac{U_1 - V_2}{V_1}\theta - \frac{U_2}{V_1} = 0 = (\theta - a_1)(\theta - a_2):$$

in this case (c) becomes $d_t u + Au + B = 0$;

by integrating which we obtain the primitive equations

$$y + a_1 x = \epsilon^{-a_1 t} \textstyle\int_t \epsilon^{a_1 t} b_1,$$

$$y + a_2 x = \epsilon^{-a_2 t} \textstyle\int_t \epsilon^{a_2 t} b_2;$$

a_1, b_1, a_2, b_2, being the values of A and B when a_1, a_2 are substituted for θ.

If we have two equations each containing three variables, we may by elimination obtain three equations of the form

$$d_t x + M_1 x + N_1 y + P_1 z + Q_1 = 0, \qquad (a)$$

$$d_t y + M_2 x + N_2 y + P_2 z + Q_2 = 0, \qquad (b)$$

$$d_t z + M_3 x + N_3 y + P_3 z + Q_3 = 0, \qquad (c)$$

in $(a) + (b) \times \theta_1 + (c) \times \theta_2 = 0$ put $u = x + \theta_1 y + \theta_2 z$, then by making the coefficients of y and z respectively equal to nothing we obtain

$$d_t\theta_1 + (M_1 + M_2.\theta_1 + M_3.\theta_2)\theta_1 - (N_1 + N_2.\theta_1 + N_3.\theta_2) = 0,$$

$$d_t\theta_2 + (M_1 + M_2.\theta_1 + M_3.\theta_2)\theta_2 - (P_1 + P_2.\theta_1 + P_3.\theta_2) = 0,$$

whence θ_1 and θ_2 may be determined; substituting their values in

$$d_t u + (M_1 + M_2.\theta_1 + M_3.\theta_2)u + Q_1 + Q_2.\theta_1 + Q_3.\theta_2 = 0,$$

u, and thence the primitive equations, may be obtained.

If the coefficients M_1, M_2, &c. are constant, the primitive equations may be determined as in the preceding instance: the same method of investigation may be applied to a greater number of equations, and to an equation of any order, in which none of the differential coefficients exceed the first degree. (*L. D. C.* 387—91; *L. C. D.* 622—4; *Tr. L.* 286, 7.)

APPROXIMATE INTEGRATION OF DIFFERENTIAL EQUATIONS.

(49.) Assume the primitive equation to be

$$y = a_1 x^{n_1} + a_2 x^{n_2} + a_3 x^{n_3} + \text{\&c.}$$

the exponents n_1, n_2, &c. being an ascending order,

then $d_x y = n_1 a_1 x^{n_1 - 1} + n_2 a_2 x^{n_2 - 1} + \text{\&c.}$

$$d_x^2 y = n_1 (n_1 - 1) a_1 x^{n_1 - 2} + n_2 (n_2 - 1) a_2 x^{n_2 - 2} + \text{\&c.}$$

&c. = &c.

By substituting these values of y, $d_x y$, &c. in the proposed differential equation, and equating corresponding indices and coefficients, we may obtain as many terms of the series as we please: the law both of the indices and coefficients is frequently evident from two or three terms.

(*L. D. C.* 392, 3: *L. C. D.* 660—7.)

(50.) *Approximation by a continued fraction.*

Assume the primitive equation to be

$$y = \frac{a_1 x^{n_1}}{1 +} \, \frac{a_2 x^{n_2}}{1 +} \, \frac{a_3 x^{n_3}}{1 + \text{\&c.}},$$

and substitute $a_1 x^{n_1}$, $n_1 a_1 x^{n_1 - 1}$, &c. for $y, d_x y$, &c. respectively in the given differential equation; then by a comparison of corresponding terms, a_1 and n_1 may be determined. Then

assume $$y = \frac{a_1 x^{n_1}}{1 + a_2 x^{n_2}},$$

and for y, $d_x y$, &c. substitute their values obtained from this equation, by which means a_2 and n_2 may be determined; the same process may be continued as far as we please.

(*L. D. C.* 394, 5; *L. C. D.* 668—71.)

(51.) If we have an equation of the form

$$d_x^n y = f(x, y, d_x y, \text{\&c. } d_x^{n-1} y), \text{ then}$$

$$y = (y)_{x=a} + d_{x=a} y \cdot \frac{x-a}{1} + d_{x=a}^2 y \frac{(x-a)^2}{1.2} + \text{\&c.} + d_{x=a}^{n-1} \frac{(x-a)^{n-1}}{1.2 \ldots (n-1)}$$

$$+ f(a, (y)_{x=a}, d_{x=a} y, \text{\&c. } d_{x=a}^{n-1} y) \frac{(x-a)^n}{1.2.3 \ldots n} + \text{\&c.}$$

This series may always be made covergent by taking a value of x sufficiently near to a. (*L. C. D.* 659.)

(52.) *Lagrange's Method.* Let $y = \phi(x, c_1, c_2, \&c.\ c_n)$, in which c_1, c_2, &c. are arbitrary constants, be the complete integral of

$$d_x^n y + u = 0:$$

in order to extend this result to the equation

$$d_x^n y + u = av,$$

in which u and v are functions of x, y, $d_x y$, &c. $d_x^{n-1} y$, the quantities c_1, c_2, &c. c_n, must be considered variable; n equations may in this case be obtained, of the form

$$d_{c_1} y . d_x c_1 + d_{c_2} y . d_x c_2 + \&c. + d_{c_n} y . d_x c_n = 0,$$

$$d_{c_1} d_x y . d_x c_1 + d_{c_2} d_x y . d_x c_2 + \&c. + d_{c_n} d_x y . d_x c_n = 0,$$

$$\&c. \qquad\qquad \&c.$$

$$d_{c_1} d_x^{n-1} y . d_x c_1 + d_{c_2} d_x^{n-1} y . d_x c_2 + \&c. + d_{c_n} d_x^{n-1} y . d_x c_n = av.$$

From these equations the values of $d_x c_1$, $d_x c_2$, &c. $d_x c_n$ may be obtained

$$\leftrightharpoons d_x c = a T . v, \qquad (1)$$

T_1, T_2, &c. being functions of x, c_1, c_2, &c. which being constant when $a = 0$, may be expanded in the form

$$c_1 = C_1 + a z_1 + a^2 z_1' + \&c.$$
$$c_2 = C_2 + a z_2 + a^2 z_2' + \&c.$$
$$\&c. = \&c.$$
$$c_n = C_n + a z_n + a^2 z_n' + \&c.$$

these values being substituted in the equations (1), an approximation may be obtained by neglecting all powers of a superior to the first.

(*L. C. D.* 674, 5; *Lagrange, Berlin Mem. an.* 1781.)

COMPARISON OF ELLIPTIC TRANSCENDENTS.

(53.) Let the given equation be $(a + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4)^{-\frac{1}{2}}$

$$+ (a + a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4)^{-\frac{1}{2}} d_x y = 0.$$

Suppose x and y to be functions of a new variable t, such that

$$d_t x = + (a + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4)^{\frac{1}{2}},$$
$$d_t y = - (a + a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4)^{\frac{1}{2}};$$

differentiating, and adding the results, we obtain

$$d_t^2 p = a_1 + a_2 p + \tfrac{3}{4} a_3 (p^2 + q^2) + \tfrac{1}{2} a_4 p (p^2 + 3q^2), \qquad (1)$$

$\frac{1}{2}(p+q)$ and $\frac{1}{2}(p-q)$ being substituted for x and y respectively; also

$$d_t p . d_t q = a_1 q + a_2 pq + \tfrac{1}{4} a_3 q (3p^2 + q^2) + \tfrac{1}{2} a_4 pq (p^2 + q^2): \qquad (2)$$

$(1) \times q - (2)$ gives by integration

$$\overline{a + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4}\Big|^{\frac{1}{2}} - \overline{a + a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4}\Big|^{\frac{1}{2}}$$
$$= (x - y) . \overline{c^2 + a_3 (x + y) + a_4 (x + y)^2}\Big|^{\frac{1}{2}}.$$

in which c is an arbitrary constant.

Another integral may be obtained in a similar manner, by substituting in (2) the value of $d_t p$, whence

$$d_t q = \frac{a_1 + a_2 p + \frac{1}{4} a_3 (3p^2 + q^2) + \frac{1}{2} a_4 p (p^2 + q^2)}{(c^2 + a_3 p + a_4 p^2)^{\frac{1}{2}}}.$$

If the quantity $a + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$ is a perfect square, no result will be obtained; but by assuming

$$\overline{a + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4}\Big|^{\frac{1}{2}} = \overline{(b + cx + ex^2)^2 + k}\Big|^{\frac{1}{2}},$$

and in the development, neglecting the powers of k superior to the first, the above integral may be reduced to that of the equation

$$\frac{1}{a + bx + cx^2} + \frac{d_x y}{a + by + cy^2} = 0.$$

(54.) Let the given equation be

$$\frac{1}{\{1 - (e \sin x)^2\}^{\frac{1}{2}}} + \frac{d_x y}{\{1 - (e \sin y)^2\}^{\frac{1}{2}}} = 0.$$

Assume $d_t x = \{1 - (e \sin x)^2\}^{\frac{1}{2}}, \quad d_t y = \{1 - (e \sin y)^2\}^{\frac{1}{2}}$, then by a method analogous to the preceding we obtain

$$\overline{1 - (e \sin x)^2}\Big|^{\frac{1}{2}} - \overline{1 - (e \sin y)^2}\Big|^{\frac{1}{2}} = c_1 \sin (x - y),$$

or $$\overline{1 - (e \sin x)^2}\Big|^{\frac{1}{2}} + \overline{1 - (e \sin y)^2}\Big|^{\frac{1}{2}} = c_2 \sin (x + y);$$

either of which may be the required integral; or if $\varkappa$ represents the value of x when $y = 0$, the integral assumes the form

$$\cos x . \cos y - \sin x . \sin y . \overline{1 - (e \sin \varkappa)^2}\Big|^{\frac{1}{2}} = \cos \varkappa.$$

(*L. C. D.* 690—711.)

(55.) The most general form of a total differential equation containing three variables is

$$M + Nd_x y + Pd_x z = 0: \qquad (1)$$

if this equation answers the criterion of integrability (Art. 28.) its integral may be obtained under the form

$$\phi(x,y,z) = c.$$

If (1) is not immediately integrable, but may be rendered so by the introduction of a factor Q, then

$$P(d_y M - d_x N) - N(d_z M - d_x P) + M(d_z N - d_y P) = 0.$$

If this equation is satisfied, the given equation may be thus integrated: let $U = 0$ be the integral of $QM + QNd_x y = 0$, then $U + Z = 0$ will be the complete integral. Z may be found by differentiating and comparing the result with the given equation.

If in the proposed equation the quantities $d_x y$, $d_x z$ exceed the first degree, these methods are applicable only when it may be resolved into factors of the form (1).

(*L. D. C.* 414—7; *L. C. D.* 713—21.)

If the equation (1) is not integrable by the above method, a solution may be thus obtained: let Q be the factor which renders the equation $M + Nd_x y$ integrable, and let the integral be $U = c$, z being considered constant. By differentiating this last equation considering x,y,z, and c, all variable, and comparing the result with (1) × Q, we obtain the equation

$$d_z U - QP = d_z c:$$

putting $c = \phi(z)$, and $d_z c = \phi'(z)$, we have

$$U = \phi(z),$$

$$d_z U - QP = \phi'(z)$$

which system of equations will satisfy the proposed equation (1) whatever be the form of the function ϕ.

If the equation contain four variables, a solution may be obtained by an analogous method.

(*L. D. C.* 418—21; *L. C. D.* 808—13.)

PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER.

(56.) The simplest class of partial differential equations consists of those which contain only one differential coefficient of the first order: let the proposed equation be

$$d_x z = f(x,y),$$

then $$z = \int_x f(x,y) + \phi(y).$$

Let the equation be $d_x z = f(x,y,z)$,

and let $u = c$ be the integral of this equation, y being considered constant; then the required integral is

$$u = \phi(y).$$

If any other variables are contained in f, they must likewise be included in the function ϕ.

(57.) The most general equation containing two partial differential coefficients of the first order is

$$Pd_x z + Qd_y z + R = 0.$$

Since $$d_x(z)^* = d_x z + d_y z . d_x y, \qquad (1)$$

by eliminating $d_x z$, we obtain

$$(Pd_x y - Q)\, d_y z = Pd_x(z) + R.$$

If P and Q contain x and y only, and P and R, x and z only, then the equations

$$Pd_x y - Q = 0, \qquad Pd_x z + R = 0,$$

may respectively be rendered integrable by the factors M, N; let the integrals obtained from these equations be

$$U = 0, \qquad V = 0,$$

then if we assume $\dfrac{Nd_y z}{M} = \phi'(U)$, the required integral will be

$$V = \phi(U).$$

If the two equations $U = a$, $V = b$, may be obtained by integrating conjointly any two of the equations

---

* For an explanation of this notation, see *Tr. L.* Art. 126.

$$Pd_xy - Q = 0, \quad Pd_xz + R = 0, \quad Qd_yz + R = 0, \qquad (2)$$

containing x, y, and z, then $V = \phi(U)$ will be the required integral.

When P, Q, and R are functions of all three variables, the integration of (2) cannot generally be effected; an integral may however sometimes be obtained by means of the following formulæ derived from (1);

$$z = xd_xz + \int_y (d_yz - xd_xd_yz),$$
$$= yd_yz + \int_x (d_xz - yd_xd_yz).$$

(58.) The integration of partial differential equations of the first order may frequently be effected by the introduction of an indeterminate quantity: let the proposed equation be reduced to the form

$$f_1(d_xz, x) = f_2(d_yz, y);$$

assume $f_1(d_xz, x) = u = f_2 d_yz, y)$,

whence $\quad d_xz = \phi_1(u, x), \quad d_yz = \phi_2(u, y)$:

let $\quad \int_x \phi_1(u, x) = P, \quad$ and $\int_y \phi_2(u, y) = Q$,

from these equations we may obtain

$$z + \psi(u) = P + Q,$$
$$d_u\psi(u) = d_uP + d_uQ;$$

from which equations the integral of the proposed equation may be obtained: a similar method of introducing an arbitrary quantity may be applied to some other equations.

(59.) If in the equation $Pd_xz + Qd_yz + R = 0$,

P, Q, and R are homogeneous functions of x, y, and z, and of n dimensions, assume $\quad x = uz, \quad y = vz$,

and let $\quad P = P_1z^n, \quad Q = Q_1z^n, \quad R = R_1z^n$,

then from (2) we obtain

$$(P_1 + uR_1)d_uv - (Q_1 + vR_1) = 0,$$

which may be integrated, since P_1, Q_1, and R_1 contain only

u and v: eliminating u or v from the equation thus found, and one of the equations

$$(P_1 + R_1 u) d_u z + R_1 z = 0,$$

$$(Q_1 + R_1 v) d_v z + R_1 z = 0,$$

the resulting equation may be integrated, and the values of u and v being substituted, the required integral will be obtained.

(60.) If we have an equation of the form

$$d_x z = f(d_y z, x, y, z),$$

by substituting this value of $d_x z$ in the equation

$$d_x(z) = d_x z + d_y z . d_x y,$$

we obtain $$d_x(z) = f(d_x z, x, y, z) + d_y z . d_x y.$$

Let u be such a function of $d_x z$ as, being considered constant, will render this equation integrable, and let the integral be

$$\phi(u, x, y, z) + \psi(u) = 0,$$

the function ψ being such that

$$d_u \phi(u, x, y, z) + d_u \psi(u) = 0:$$

the primitive equation may be obtained by eliminating u from these equations.

(*L. D. C.* 422—35; *L. C. D.* 728—34; *Tr. L.* 311—5.)

(61.) A method analogous to that of (Art. 57.) may be applied to equations containing more than three variables: let the proposed equation be

$$N d_u z + P d_x z + Q d_y z + R = 0,$$

then since $$d_u(z) = d_u z + d_x z . d_u x + d_y z . d_u y,$$

by eliminating $d_u z$ we obtain

$$N d_u(z) + R = d_x z (N d_u x - P) + d_y z (N d_u y - Q).$$

If the equations

$$N d_u z + R = 0, \quad N d_u x - P = 0, \quad N d_u y - Q = 0,$$

contain only u and z, u and x, u and y respectively, they may be rendered integrable by the factors M_1, M_2, M_3: let the integrals obtained from these be $V=0$, $T=0$, $U=0$, then if

$$\frac{M_3}{M_1} d_x z = \phi'(T), \qquad \frac{M_3}{M_2} d_y z = \psi'(U)$$

the required integral will be

$$V = \phi(T) + \psi(U). \qquad (L.\ C.\ D.\ 735.)$$

For the investigation of partial differential equations in which the differential coefficients exceed the first degree, see (*L. C. D.* 740—9.)

PARTIAL DIFFERENTIAL EQUATIONS OF SUPERIOR ORDERS.

(62.) Differential equations of the form

$$f(x, y,\ d_y^n z,\ d_x d_y^n z, \ \&\text{c.}\ d_x^m d_y^n z) = 0$$

may, by assuming $d_y^n z = u$, be reduced to

$$f(x, y, u,\ d_x u, \ \&\text{c.}\ d_x^m u) = 0.$$

All equations of this form, in which the differential coefficients involve only two variables may be considered as equations containing only two variables, and integrated by preceding methods: arbitrary functions of all other variables contained in the function f must however be substituted for arbitrary constants in the integrals thus obtained.

The following are some of the simplest forms belonging to this class of equations.

[1] $d_x^2 z + f(x,y) = 0;$

$$z + \textstyle\int_x^2 f(x,y) + x \cdot \phi(y) + \psi(y) = 0.$$

[2] $d_x^2 z + d_x z . f(x,y) = 0;$

$$z + \textstyle\int_x e^{\int_x f(x,y) + \phi(y)} + \psi(y) = 0.$$

[3] $d_x^2 z + f(x,y) d_x z + f_1(x,y) = 0;$

$$z + \textstyle\int_x e^{-\int_x f(x,y)} \left\{ \int_x e^{\int_x f(x,y)} f_1(x,y) + \phi(y) \right\} + \psi(y) = 0.$$

[4] $d_x d_y z + f(x,y) d_x z + f_1(x,y) = 0;$

$$z + \textstyle\int_x e^{-\int_y f(x,y)} \left\{ \int_y e^{\int_y f(x,y)} + \phi(y) \right\} + \psi(x) = 0.$$

(63.) The most general partial differential equation of the second order is

$$Rd_x^2 z + Sd_x d_y z + Td_y^2 z + V = 0; \qquad (1)$$

in which R, S, &c. are functions of $d_x z$, $d_y z$, x, y, and z. let $d_x z$, $d_y z$, $d_x^2 z$, $d_x d_y z$, $d_x^2 z$; be represented by

p, q, r, s, t, respectively;

then (1) becomes $Rr + Ss + Tt + V = 0$;

also

$$d_x(p) = r + sd_x y,$$

$$d_x(q) = s + td_x y:$$

eliminating r and t between these three equations, and putting $d_x y = m$, we obtain

$$Rmd_x(p) + Td_x(q) + Vm = s\{Rm^2 - Sm + T\}$$

which must be satisfied independently of s; let $M = a$, $N = b$ be two equations which satisfy

$$Rmd_x p + Td_x q + Vm = 0, \qquad (2)$$

$$Rm^2 - Sm + T = 0; \qquad (3)$$

then $N = \phi(M)$ is the required integral.

If R, S, and T, are constant, the values of M are constant, let them be a_1, a_2; from (2), (3) we obtain

$$Ra_1 p + Tq + a_1 \textstyle\int_x V = f(y - a_1 x),$$

$$Ra_2 p + Tq + a_2 \textstyle\int_x V = f(y - a_2 x);$$

eliminating p between either of these equations and

$$d_x(z) = p + qd_x y,$$

and integrating the result, we obtain an equation of the form

$$Rz = \textstyle\int_x^2 V + \phi(y - a_1 x) + \psi(y - a_2 x).$$

In determining $\int_x^2 V$, $a_1 x + b_1$ must be substituted for y in V, and $y - a_1 x$ for b_1 after the first integration; then for y substitute $a_2 x + b_2$, and $y - a_2 x$ for b_2 after the second integration.

The equation to vibrating chords is $r - c^2 t = 0$, in this case

$$a_1 = c, \qquad a_2 = -c, \qquad R = 1, \qquad \text{and} \int_x^2 V = 0;$$

and

$$z = \phi(y - cx) + \psi(y + cx).$$

(64.) The introduction of an arbitrary function is in some cases applicable to equations of the second order; suppose we have the general equation to developable surfaces,

$$rt - s^2 = 0; \qquad (L.\ C.\ D.\ 339.)$$

let $\frac{r}{s} = u = \frac{s}{t}$, then $r = su$, and $s = tu$,

whence $r + s d_x y = u(s + t d_x y)$, or $d_x(p) = u . d_x(q)$; the integral of this equation is $p = \phi(q)$, and therefore

$$d_x(z) = \phi(q) + q d_x y;$$

by the integration of which we obtain

$$z = x\phi(q) + qy + f(q),$$

also $$0 = x\phi'(q) + y + f'(q):$$

from which equations the value of z may be obtained.

(*L. D. C.* 436—46; *Tr. L.* 316—20.)

(65.) In the equation

$$Rr + Ss + Tt + Pp + Qq + Nz = M$$

let the coefficients R, S, &c. be functions of x and y only. Putting $d_x y = m$, as in Art. (63.) we obtain

$$d_x y - m = 0,$$

$$Rm d_x(p) + T d_x(q) + Pp + Qq + Nz - m = 0:$$

the value of y in terms of x obtained from the first of these equations being substituted in the second will reduce it to an equation containing four variables, x, z, p, and q; and this being integrated simultaneously with

$$d_x(z) = p + mq,$$

an integral may be obtained

[illegible] $$Kp + Lq + Fz + G = c.$$

If the coefficients R, S, &c. are constant, then the first integrals of the proposed equation are

$$\epsilon^{n_1 x}\{m_1 Rp + Tq + m_1(P - n_1 R)z\} - m_1 \int_x M_1 \epsilon^{n_1 x} = \phi(y - m_1 x),$$

$$\epsilon^{n_2 x}\{m_2 Rp + Tq + m_2(P - n_2 R)z\} - m_2 \int_x M_2 \epsilon^{n_2 x} = \phi(y - m_2 x),$$

where M_1 and M_2 are the values of M, and n_1, n_2, those of

$$\frac{m(mP-Q)}{Rm^2-T},$$

corresponding to the two roots of the equation

$$Rm^2-Sm+T=0. \qquad (L.\ C.\ D.\ 756,\ 7.)$$

(66.) The most general form of a partial differential equation of the third order is

$$Ad_x^3z+Bd_x^2d_yz+Cd_xd_y^2z+Ed_y^3z=V,$$

in which A, B, &c. are functions of x, y, z, and the differential coefficients of the first and second order. By eliminating d_x^3z, &c. between this equation and

$$d_x(r)=d_x^3z+d_x^2d_yz.d_xy,$$
$$d_x(s)=d_x^2d_yz+d_xd_y^2z.d_xy,$$
$$d_x(t)=d_xd_y^2z+d_y^3z.d_xy,$$

and putting $d_xy=m$, we obtain

$$d_xy-m=0,$$
$$Ad_x(r)+(B-Am)d_x(s)+(C-Bm+Am^2)d_x(t)-V=0,$$

in which the value of m depends on the equation

$$Am^3-Bm^2+Cm-E=0:$$

If from these we can obtain two primitive equations $M=a$, $N=b$, then $N=\phi(M)$ will be one of the required first integrals.

When A, B, C, and E, are constant, and V a function of x and y only, then the three first integrals of the proposed equation will be

$$Ar+(B-Am_1)s+(C-Bm_1+Am_1^2)t-\int_x V_1=\phi(y-m_1x),$$
$$Ar+(B-Am_2)s+(C-Bm_2+Am_2^2)t-\int_x V_2=\chi(y-m_2x),$$
$$Ar+(B-Am_3)s+(C-Bm_3+Am_3^2)t-\int_x V_3=\psi(y-m_3x);$$

If $V=0$, the primitive equation will be

$$z=\phi(y-m_1x)+\chi(y-m_2x)+\psi(y-m_3x).$$

If two of the roots, as m_1 and m_2 are equal, the corresponding part of the value of z will be

$$\phi(y-m_1x)+x\chi(y-m_1x).$$

The same method of investigation may be extended to equations of any order, in which the partial differential coefficients appear only in the first degree. (*L. C. D.* 758–61.)

(67.) *Reduction of partial differential equations of superior orders to the first order.*

Let x and y be supposed to be functions of some new variables u and v, then

$$d_x(u) = d_x u + d_y u \,.\, d_x y,$$
$$d_x(v) = d_x v + d_y v \,.\, d_x y,$$
$$d_x(z) = d_u z \,.\, d_x u + d_v z \,.\, d_x v;$$

hence $\quad p = d_u z \,.\, d_x u + d_v z \,.\, d_x v, \quad q = d_u z \,.\, d_y u + d_v z \,.\, d_y v$: by differentiating these equations we obtain

$$r = d_u^2 z \,.\, (d_x u)^2 + 2 d_u d_v z \,.\, d_x u \,.\, d_x v + d_v^2 z \,.\, (d_x v)^2 \\ + d_u z \,.\, d_x^2 u + d_v z \,.\, d_x^2 v,$$

$$s = d_u^2 z \,.\, d_x u \,.\, d_y u + d_u d_v z \,.\, d_x u \,.\, d_y v + d_u d_v z \,.\, d_y u \,.\, d_x v \\ + d_v^2 z \,.\, d_x v \,.\, d_y v + d_u z \,.\, d_x d_y u + d_v z \,.\, d_x d_y v,$$

$$t = d_u^2 z \,.\, (d_y u)^2 + 2 d_u d_v z \,.\, d_y u \,.\, d_y v + d_v^2 z \,.\, (d_y v)^2 \\ + d_u z \,.\, d_y^2 u + d_v z \,.\, d_y^2 v.$$

By substituting these values in the equation

$$Rr + Ss + Tt + Pp + Qq + Nz = M,$$

we obtain

$$\{R(d_x u)^2 + S d_x u \,.\, d_y u + T(d_y u)^2\} d_u^2 z \\ + \{2R d_x u \,.\, d_x v + S(d_x u \,.\, d_y v + d_y u \,.\, d_x v) + 2T d_y u \,.\, d_y v\} d_u d_v z \\ + \{R(d_x v)^2 + S d_x v \,.\, d_y v + T(d_y v)^2\} d_v^2 z \\ + \{R d_x^2 u + S d_x d_y u + T d_y^2 u + P d_x u + Q d_y u\} d_u z \\ + \{R d_x^2 v + S d_x d_y v + T d_y^2 v + P d_x v + Q d_y v\} d_v z \\ + Nz = M. \qquad (1)$$

The relations between u, v, x, and y may be determined by assuming

$$R(d_x u)^2 + S d_x u \,.\, d_y u + T(d_y u)^2 = 0,$$
$$R(d_x v)^2 + S d_x v \,.\, d_y v + T(d_y v)^2 = 0;$$

which equations, by putting $d_x u = m d_y u$, $d_x v = n d_y v$, become

$$Rm^2 + Sm + T = 0,$$

$$Rn^2 + Sn + T = 0.$$

The values of u and v thus obtained being substituted in (1), the resulting equation is

$$d_u d_v z + P_1 d_u z + Q_1 d_v z + N_1 z = M_1, \qquad (2)$$

which may be put under the form

$$d_u(d_v z + P_1 z) + Q_1 \{ d_v z + \frac{1}{Q_1}(N_1 - d_u P_1) z \} = M_1;$$

and this may be reduced to an equation of the first order if

$$d_u P_1 + P_1 Q_1 - N_1 = 0; \qquad (3)$$

for by putting $d_v z + P_1 z = z_1$, (2) becomes

$$d_u z_1 + Q_1 z_1 = M_1.$$

The complete integral of (2) obtained from the integrals of these equations is

$$z = \epsilon^{-\int_v P_1} \{ \phi(u) + \int_v \epsilon^{\int_v P_1 - \int_u Q_1} [\psi(v) + \int_u M_1 \epsilon^{\int_u Q_1}] \}.$$

If the equation (3) is not satisfied, assume

$$N_1 - P_1 Q_1 - d_u P_1 = a,$$

then putting, as before, $d_v z + P_1 z = z_1$, (2) becomes

$$d_u z_1 + Q_1 z_1 + a z = M_1:$$

the values of z and $d_v z$ determined from this being substituted in the preceding equation, we have, for the determination of z_1, the equation

$$d_u d_v z_1 + P_2 d_u z_1 + Q_1 d_v z_1 + N_2 z_1 = M_2;$$

in which $P_2 = P_1 - \frac{1}{a} . d_v a,$

$$N_2 = a - \frac{Q_1}{a} d_v a + d_v Q_1 + P_1 Q_1,$$

$$M_2 = d_v M_1 - \frac{M_1}{a} d_v a + P_1 M_1.$$

The equation (2) is thus reduced to another of the same form, with which the same process may be repeated.

(*L. C. D.* 764—7.)

(68.) If we have an equation containing x, y, and z, then by Maclaurin's theorem

$$z = (z)_{x=0} + x . d_{x=0} z + \frac{x^2}{1.2} . d^2_{x=0} z + \&c. \quad {}^{\alpha}$$

If the equation is $d_x z = \phi(d_y z, x, y)$, then $(z)_{x=0}$ is an arbitrary function of y: and generally, if the equation is

$$d_x^n z = \phi(d_x^{n-1} d_y z, \&c.\ x, y),$$

then $(z)_{x=0}$, $d_{x=0} z$, &c. $d^{n-1}_{x=0} z$, are arbitrary functions of y.

If any coefficient becomes infinite, when $x = 0$, then $x + a$ may be substituted for x in the given function, and a series obtained by developing according to the powers of x.

An integral may sometimes be obtained by assuming

$$z = Y + Y_1 x + Y_2 x^2 + \&c.$$

then substituting the differential coefficients obtained from this series in the proposed equation, and equating the coefficients of the same powers of x: the form of series which it will be most convenient to adopt must be determined from the circumstances of the case. (*L. D. C.* 446—9; *L. C. D.* 778—91.)

PARTICULAR SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS.

(69.) If $U = 0$ represents the complete integral of a partial differential equation of the first order, containing the arbitrary constants a and b, the system of equations

$$U = 0, \qquad d_a U + d_b U . d_a b = 0,$$

will comprise all possible solutions of the differential equation: if however we make $d_a U = 0$, $d_b U = 0$, and substitute the values of a and b in terms of x, y, and z, obtained from these equations, in $U = 0$, we shall obtain an equation independent of a and b, and which will be a particular solution.

[α] $z = \overset{\infty}{S}_m x^{m-1} . d^{m-1}_{x=0} : z.$

The same reasoning may be applied to partial differential equations of higher orders, of which we have obtained the complete first integral: the solutions obtained will be of an order next inferior to that of the given equation.

(70.) Let it be required to determine whether a particular value $z = U$ is or is not comprised in the equation

$$z = f\{x, y, \phi(P), \psi(Q)\},$$

the general solution of the partial differential equation, $Z = 0$.

Let the general solution be put under the form

$$z = U + au,$$

in which it is necessary that u should contain as many arbitrary functions as the general solution, and also have a finite value when $a = 0$.

We have $p = d_x U + a d_x u$, $q = d_y U + a d_y u$,

$r = d_x^2 U + a d_x^2 u$, $s = d_x d_y U + a d_x d_y u$, $t = d_y^2 U + a d_y^2 u$, &c.

and since $z = U$ will satisfy the equation $Z = 0$, we obtain

$$d_z Z \,.\, u + d_p Z \,.\, d_x u + d_q Z \,.\, d_y u$$
$$+ d_r Z \,.\, d_x^2 u + d_s Z \,.\, d_x d_y u + d_t Z \,.\, d_y^2 u + \&c. = 0.$$

If this substitution renders all the differential coefficients of Z respectively equal to nothing, except $d_z Z$, this last equation cannot be satisfied unless $u = 0$, in which case no arbitrary functions can be introduced in the equation $z = U$. If $d_p Z$, and $d_q Z$ do not vanish, the function u, depending on the equation

$$d_z Z \,.\, u + d_p Z \,.\, d_x u + d_q Z \,.\, d_y u = 0,$$

will involve one arbitrary constant. (*L. C. D.* 793—5.)

APPLICATION OF THE DIFFERENTIAL AND INTEGRAL CALCULUS TO GEOMETRY.

THE CONTACT OF LINES.

(1.) Let $y=\phi(x)$, $y=\psi(x)$, be the equations of two curves having the co-ordinates x, y common to both, then

$$\phi(x+Dx)=y+\frac{Dx}{1}d_x\phi(x)+\frac{(Dx)^2}{1\,.\,2}d_x^2\phi(x)+\&c.$$

$$\psi(x+Dx)=y+\frac{Dx}{1}d_x\psi(x)+\frac{(Dx)^2}{1\,.\,2}d_x^2\psi(x)+\&c.$$

If $d_x\phi(x)=d_x\psi(x)$, they have a *contact of the first order*, and do not intersect each other. If $d_x^2\phi(x)=d_x^2\psi(x)$, they have a contact of the second order, and intersect each other at the point (x, y): and generally, if $d_x^n\phi(x)=d_x^n\psi(x)$, they have a contact of the nth order, and intersect or not, according as n is even or odd.

Let x, y, be the co-ordinates of a curve, x_1, y_1, those of its tangent, then the equation of the tangent is

$$y_1-y=d_xy\,(x_1-x).$$

The subtangent $=-\dfrac{y}{d_xy}$.

The distances from the origin at which the tangent cuts the axes of x and y respectively are

$$x-\frac{y}{d_xy},\qquad y-xd_xy.$$

The equation of the normal is

$$y_1-y=-\frac{1}{d_xy}(x_1-x).$$

The subnormal $= y d_x y$.

The part of the normal intercepted between the axis and the curve $= y . \overline{1 + (d_x y)^2}|^{\frac{1}{2}}$.

(2.) The *circle of curvature*, or *osculating circle*, is one which has a contact of the second order with any proposed curve. Let ρ be its radius, x_1, y_1, the co-ordinates of its centre, then

$$x_1 = x - \frac{d_x y}{d_x^2 y}\{1 + (d_x y)^2\}, \qquad y_1 = y + \frac{1}{d_x^2 y}\{1 + (d_x y)^2\},$$

$$\rho = - \frac{1}{d_x^2 y} . \overline{1 + (d_x y)^2}|^{\frac{3}{2}}.$$

If x and y are functions of a third variable, t, then

$$\rho = \frac{\{(d_t x)^2 + (d_t y)^2\}^{\frac{3}{2}}}{d_t y . d_t^2 x - d_t x . d_t^2 y}.$$

If the arc s is considered the independent variable, then

$$\rho = \frac{d_s y}{d_s^2 x} = - \frac{d_s x}{d_s^2 y},$$

$$= \{(d_s^2 y)^2 + (d_s^2 x)^2\}^{-\frac{1}{2}}.$$

(*Laplace, Mec. Cel.* Liv. 1. Ch. ii.)

When the radius of curvature is a maximum or minimum, there is a contact of the third order, and in that case the circle of curvature does not cut the curve.

The curve $y = \phi(x)$ is convex or concave towards the axis of x, according as $\phi(x + Dx)$ and $d_x^2 y$ have the same or different signs. (*L. D. C.* 151.)

(3.) The *evolute* of a plane curve is the locus of all centres of curvature: its equation may be found by substituting for $d_x y$ and $d_x^2 y$ their values obtained from the equation of the proposed curve, $\phi(x,y) = 0$, in the equations

$$x_1 = x - \frac{d_x y}{d_x^2 y}\{1 + (d_x y)^2\}, \qquad y_1 = y + \frac{1}{d_x^2 y}\{1 + (d_x y)^2\},$$

and eliminating x and y between these, and $\phi(x,y) = 0$: the resulting equation between x_1 and y_1 will be the equation of the evolute.

(4.) Let $y = a_r x^{n_r} + a_{r-1} x^{n_{r-1}} + \&\text{c.} + a_1 x^{n_1} + a x^n$
$+ a_{-1} x^{n_{-1}} + a_{-2} x^{n_{-2}} + \&\text{c.}$
be the equation of a plane curve, in which the quantities n_r, n_{r-1}, &c. form a decreasing series, of which n_{-1} is the first negative term, then the curve whose equation is

$$y = a_r x^{n_r} + a_{r-1} x^{n_{r-1}} + \&\text{c.} + a_1 x^{n_1} + a x^n,$$

is an *asymptote* to the former.

If $n_r = 1$, and $a_{r-1}, \ldots a$ respectively $= 0$, the asymptote is a right line, and the corresponding infinite branch is of the *hyperbolic* kind: for all other values of n_r, &c. a_r, &c. the asymptote and curve are of the *parabolic* kind.

(5.) *Methods of expanding* y *in a descending series in terms of* x.

[1] Assume $u = \frac{y}{x} = yz$, then by Maclaurin's theorem

$$u = (u)_{z=0} + z.d_{z=0}u + \frac{z^2}{1.2}.d^2_{z=0}u + \&\text{c.}$$

or
$$y = x.(u)_{x=\infty} + d_{x=\infty}u + \frac{1}{1.2.x}.d^2_{x=\infty}u + \&\text{c.}\ [a]$$

[2] Assume $y = ax^m$, and after substituting this value for y, equate the indices of any two terms; then if the other terms, when the value obtained for m has been substituted, are of less dimensions than these, we shall obtain the first term of a descending series. By assuming

$$y = ax^m + bx^n,$$

a second term may be obtained, and so on: this method is generally very tedious, if applied to an equation consisting of more than four terms.

A rectilinear asymptote may frequently be conveniently obtained by finding the value of $x - \frac{y}{d_x y}$, and $y - xd_x y$, when $x = \infty$; if either or both of these quantities be finite, there is an asymptote, the direction of which will be determined by the value of the quantity $d_{x=\infty}y$.

---

[a] $y = \overset{\infty}{\mathrm{S}}_m x^{2-m}.d^{m-1}_{z=\infty}:u.$

SINGULAR POINTS.

(6.) Let the equation of a curve, and its differential be respectively $\phi(x,y)=0, \quad M+Nd_xy=0,$ then the values of x and y which satisfy the equations

$$\phi(x,y)=0, \qquad M=0,$$

but do not satisfy $N=0$, give a maximum or minimum ordinate, according as d_x^2y is negative or positive: and those values which satisfy $\phi(x,y)=0, \quad N=0,$ but not $M=0$, give a maximum or minimum abscissa.

If $d_x^2y=0$, (x,y) is a point of *inflexion*, or *contrary flexure*, at which point the tangent cuts the curve: if however $d_x^3y=0$ is satisfied by the same values of x and y, (x,y) is a point of *undulation*, at which point the tangent merely meets the curve.

Generally, if several successive differential coefficients, as far as the n^{th}, vanish for the same values of x and y, the curve has a contact of the n^{th} order with its tangent, which intersects the curve or meets it, according as n is even or odd.

(7.) If the same values of x and y satisfy the equations

$$\phi(x,y)=0, \qquad M=0, \qquad N=0,$$

then $d_xy=\frac{0}{0}$, in which case the values of d_xy may be found by preceding methods. (*Diff. Calc.* 15.)

If the equation $d_xy=\frac{0}{0}$ has two or more unequal possible roots, then (x,y) is a *multiple point*, through which as many branches of the curve pass as the above equation has possible roots.

If all the roots of $d_xy=\frac{0}{0}$ are impossible, then (x,y) is an insulated or *conjugate point*.

If the same values of x and y render $d_x^2y=\frac{0}{0}$, the roots of this equation may be determined as above: if it has two or more unequal possible roots, (x,y) is a *point of osculation;*

the branches which pass through this point have a common tangent: if all the roots are impossible, there is, as before, a conjugate point.

(8.) *Singular points at which* y *or any of its differential coefficients become infinite.*

Let $x + Dx$ be substituted for x, and let the corresponding value of y be expanded in a series of ascending powers of Dx, then $$E_x y = y + a_1 (Dx)^{m_1} + a_2 (Dx)^{m_2} + \&c.$$ and suppose that neither of the indices is a fraction having an even denominator: then if the numerator of m_1 is odd, there is a point of contrary flexure, the tangent at that point being parallel to the axis of y, or x, according as $m_1 >$ or < 1.

If the numerator of m_1 is even, there is a maximum or minimum ordinate, if $m_1 > 1$; and which is also a *point of regression*, or *cusp*, of the *first species*, if $m_1 < 1$.

If $m_1 = 1$, and the numerator of m_2 is odd, there is a point of contrary flexure; the tangent at that point being inclined to the axis at a finite angle.

If the denominator of m_1 alone is even, there is a minimum abscissa, and the point is a cusp of the first species, if $m_1 > 1$: if m_2 alone has an even denominator, there is a cusp of the first species, but no maximum or minimum ordinate: if m_3 alone, or any succeeding index, has an even denominator, there is a cusp of the second species.

If some terms are rendered impossible by $+Dx$, and some by $-Dx$, or if any terms become impossible for all values of Dx, there is a conjugate point. (*L. D. C.* 160—7.)

CURVES REFERRED TO POLAR CO-ORDINATES.

(9.) Let p be the perpendicular on the tangent from the pole, then $$p = \frac{u^2}{\overline{u^2 + (d_\theta u)^2}|^{\frac{1}{2}}}.$$

$$\text{The subtangent} = \frac{u^2}{d_\theta u} = \frac{rp}{(r^2 - p^2)^{\frac{1}{2}}}.$$

$$\text{The radius of curvature} = \frac{\overline{u^2 + (d_\theta u)^2}|^{\frac{3}{2}}}{u^2 + 2(d_\theta u)^2 - u . d_\theta{}^2 u} = u . d_p u.$$

$$\tfrac{1}{2}\text{the chord through the pole} = \frac{u\{u^2 + (d_\theta u)^2\}}{u^2 + 2(d_\theta u)^2 - u . d_\theta{}^2 u} = p . d_p u.$$

(*L. C. D.* 253.)

(10.) *The evolute of a spiral.* Let U be the radius vector, and P the perpendicular on the tangent in the evolute, then

$$\begin{aligned} U^2 &= u^2\{1 + (d_\theta u)^2\} - 2pu . d_\theta u, \\ P^2 &= u^2 - p^2, \qquad\qquad (1) \\ U^2 - P^2 &= (u d_\theta u - p)^2. \end{aligned}$$

When the equation between u and p is given, that between U and P may be found by eliminating u, $d_\theta u$, and p, between the given equation, and two of the equations (1); the equation thus obtained is the equation of the evolute.

When the equation between U and P is given, then that between u, $d_\theta u$, and p may be found by eliminating U and P: the curve represented by this equation is the *involute* of the given curve.

(11.) If, when $u = \infty$, θ and the subtangent have finite values, the curve admits of a rectilinear asymptote, which may be determined by drawing u at the given finite angle θ, and making the subtangent equal to the given value; then a parallel to u through the extremity of the subtangent will be the asymptote required.

If, when $\theta = \infty$, u has a finite value, the spiral admits of an *asymptotic circle*, of which that value of u is the radius.

If $d_u p = 0$, there is a point of contrary flexure.

CIRCUMSCRIBING FIGURES.

(12.) *Method of determining the maximum inscribed, or minimum circumscribing figure.*

Let $y = f(x)$ be the equation of the given figure,

$y_1 = \phi(a,b,x)$ . . . . figure sought, then $f(x) = \phi(a,b,x)$, and $f'(x) = \phi'(a,b,x)$; eliminating x from these equations, we obtain an equation of the form

$$\psi(a,b) = 0,$$

also since $\psi(a,b)$ is a maximum, or minimum,

$$\psi'(a,b) = 0,$$

from which equations a and b may be determined.

(13.) *Method of determining the curve which circumscribes any number of curves described according to a given law.*

Let the equation $\phi(x,y,a) = 0$ represent a system of curves depending on the value of the parameter a; then by eliminating a between the equations

$$\phi(x,y,a) = 0, \qquad d_a \phi(x,y,a) = 0,$$

we obtain an equation between x and y which is the equation of the curve required.

THE CONTACT OF SURFACES.

(14.) Let x, y, z, be the co-ordinates of a given surface,
x_1, y_1, z_1, those of the tangent plane;
α, β, γ, the angles at which the tangent plane is inclined to the planes of yz, xz, xy, respectively.

The equation of the tangent plane is

$$z_1 - z = d_x z(x_1 - x) + d_y z(y_1 - y).$$

$$(\cos \alpha)^{-2} = 1 + (d_y x)^2 + (d_z x)^2,$$
$$(\cos \beta)^{-2} = 1 + (d_x y)^2 + (d_z y)^2,$$
$$(\cos \gamma)^{-2} = 1 + (d_x z)^2 + (d_y z)^2.$$

If x_1, y_1, z_1, are the co-ordinates of the normal, its equations are

$$x_1 - x = d_x z(z_1 - z),$$
$$y_1 - y = d_y z(z_1 - z).$$

If α, β, γ, are the angles which the normal makes with the axes of x, y, z, respectively, their values are the same as above.

The length of the normal intercepted between the surface, and the plane of $xy = z\{1 + (d_x z)^2 + (d_y z)^2\}^{\frac{1}{2}}$.

(15.) Let $z = \phi(x,y)$ be the equation of a given surface, and $z_1 = \psi(x_1,y_1)$ the equation of a surface touching the former, and let $d_x z$, $d_y z$, $d_x^2 z$, $d_x d_y z$, $d_y^2 z$ be, respectively represented

by p, q, r, s, t, in the first surface,

and by p_1, q_1, r_1, s_1, t_1, in the second.

If these surfaces have a contact of the first order, then

$$z_1 = z, \qquad p = p_1, \qquad q = q_1,$$

to satisfy which, the equation $z_1 = \psi(x_1,y_1)$ must contain three arbitrary constants.

For a contact of the second order, we must also have

$$r_1 = r, \qquad s_1 = s, \qquad t_1 = t,$$

to satisfy which conditions in addition to the former, the equation $z_1 = \psi(x_1,y_1)$ must contain six arbitrary constants: hence an *osculating sphere* cannot be found generally, as a circle of curvature to a plane curve. If however we suppose a section of the surface to be made by a plane passing through the normal, λ the tangent of the angle which the projection of the section on the plane xy makes with the axis of x, and ρ the radius of the sphere osculating this section, then

$$\rho = -\frac{1 + \lambda^2 + (p + q\lambda)^2}{r + 2s\lambda + t\lambda^2}(1 + p^2 + q^2)^{\frac{1}{2}}.$$

The planes of greatest and least curvature are perpendicular to each other. Let R_1 and R_2 be the radii of greatest and least curvature, and let the co-ordinates be so transformed, that the tangent plane, and the planes of greatest and least curvature may become the planes of xy, yz, xz respectively; then if θ is the angular distance of any section from the plane xz, and ρ the radius of the sphere osculating that section,

$$\rho = \frac{R_1 . R_2}{R_1 (\cos\theta)^2 + R_2 (\sin\theta)^2}.$$

$R_1 = -\frac{1}{r}$, and $R_2 = -\frac{1}{t}$, hence

$$-\frac{1}{\rho} = r(\sin\theta)^2 + t(\cos\theta)^2.$$

(*L. D. C.* 168—78; *L. C. D.* 313—24.)

(16.) Let $y = \phi_1(x)$, $z = \phi_2(x)$, be the equations of a given curve, and $y_1 = \psi_1(x_1)$, $z_1 = \psi_2(x_1)$, those of the touching line; then in order that there may be a contact of the first order, we must have

$$\phi_1(x) = \psi_1(x), \qquad \phi_2(x) = \psi_2(x),$$
$$d_x\phi_1(x) = d_x\psi_1(x), \qquad d_x\phi_2(x) = d_x\psi_2(x);$$

for a contact of the second order we must also have

$$d_x^2\phi_1(x) = d_x^2\psi_1(x), \quad d_x^2\phi_2(x) = d_x^2\psi_2(x);$$

and so on for the superior orders.

(17.) The equations of the tangent are

$$y_1 - y = d_x y(x_1 - x), \qquad z_1 - z = d_x z(x_1 - x).$$

Let α, β, γ, be the angles which the tangent makes with the axes of x, y, z, respectively, then

$$(\cos\alpha)^{-2} = 1 + (d_x y)^2, + (d_x z)^2,$$
$$(\cos\beta)^{-2} = 1 + (d_y x)^2 + (d_y z)^2,$$
$$(\cos\gamma)^{-2} = 1 + (d_z x)^2 + (d_z y)^2.$$

The length of the tangent intercepted between the curve and the plane $xy = z\{1 + (d_z x)^2 + (d_z y)^2\}^{\frac{1}{2}}$.

The equation of the normal plane is

$$(x_1 - x) + d_x y(y_1 - y) + d_x z(z_1 - z) = 0.$$

If α, β, γ, are the inclinations of the normal plane to the planes yz, xz, xy, respectively, their values are the same as above.

The distances from the origin at which the plane cuts the axes of x, y, z, respectively are

$$x + y.d_x y + z.d_x z, \qquad y + x.d_y x + z.d_y z, \qquad z + x.d_z x + y.d_z y.$$

The equation of that plane passing through the tangent in which the curvature takes place is

$$d_x^2 y\left(\frac{z_1 - z}{x_1 - x} - d_x z\right) = d_x^2 z\left(\frac{y_1 - y}{x_1 - x} - d_x y\right).$$

Let ρ be the radius of curvature, then $\frac{1}{\rho} =$

$$\overline{(d_s y . d_s^2 z - d_s z . d_s^2 y)^2 + (d_s z . d_s^2 x - d_s x . d_s^2 z)^2 + (d_s x . d_s^2 y - d_s y . d_s^2 x)^2}\Big|^{\frac{1}{2}},$$

$$= \overline{(d_s^2 x)^2 + (d_s^2 y)^2 + (d_s^2 z)^2}\Big|^{\frac{1}{2}}.$$

(*L. D. C.* 179—86; *L. C. D.* 351—60; *L. Mec. Cel.* Liv. 1. Ch. 2.)

RECTIFICATION.

(18.) Let s be the length of the arc of the plane curve, $y = \phi(x)$, then $d_x s = \overline{1 + (d_x y)^2}\Big|^{\frac{1}{2}}$, and $s = \int_x \overline{1 + \{d_x \phi(x)\}^2}\Big|^{\frac{1}{2}}$.

The value of s between the limits $x = a$, and $x = b$, is

$$\left(\int_{x=b} - \int_{x=a}\right) \overline{1 + \{d_x \phi(x)\}^2}\Big|^{\frac{1}{2}}.*$$

(19.) Let $u = \phi(\theta)$ be the equation of the curve referred to polar co-ordinates, then

$$d_\theta s = \overline{u^2 + (d_\theta u)^2}\Big|^{\frac{1}{2}}, \text{ and } s = \int_\theta \overline{\{\phi(\theta)\}^2 + \{d_\theta \phi(\theta)\}^2}\Big|^{\frac{1}{2}}.$$

(20.) Let $y = \phi(x)$, $z = \psi(x)$, be the equations of a curve of double curvature, then

$$d_x s = \overline{1 + (d_x y)^2 + (d_x z)^2}\Big|^{\frac{1}{2}}, \text{ and } s = \int_x \overline{1 + \{d_x \phi(x)\}^2 + \{d_x \psi(x)\}^2}\Big|^{\frac{1}{2}}.$$

(21.) *Rectification of the ellipse.*

$$s = \int_x \left\{\frac{1 - e^2 x^2}{1 - x^2}\right\}^{\frac{1}{2}} = \int_x \frac{1 - e^2 x^2}{\{(1 - x^2)(1 - e^2 x^2)\}^{\frac{1}{2}}}:$$

comparing this with $\int_y \frac{R_1(y^2)}{\{(1 \pm p^2 y^2)(1 \pm q^2 y^2)\}^{\frac{1}{2}}}$, (*Int. Calc.* 17.)

we have $p = 1, \quad q = e, \quad y_1 = x\left\{\frac{1 - x^2}{1 - e^2 x^2}\right\}^{\frac{1}{2}}$,

$$p_1 = 1 + (1 - e^2)^{\frac{1}{2}}, \qquad q_1 = 1 - (1 - e^2)^{\frac{1}{2}}.$$

Assume $e_1 = \frac{q_1}{p_1}$, $p_1 y_1 = x_1$, and $s_1 = \int_{x_1} \frac{1 - e_1^2 x_1^2}{\{(1 - x_1^2)(1 - e_1^2 x_1^2)\}^{\frac{1}{2}}}$,

then $2(1 + e_1)s = 2e_1 x_1 - (1 - e_1^2)(s_1 - e_1 . d_{e_1} s_1) + 2s_1. \qquad (1)$

---

* This notation will be explained in the Appendix.

By a similar transformation, we may obtain an equation of the same form between s_1 and s_2; from these equations we have

$$\frac{e_1 x_1 + s_1 - (1+e_1)s}{1-e_1^2} = \frac{1+e_2}{2} \cdot \frac{e_2 x_2 + s_2 - (1+e_2)s_1}{1-e_2^2}. \qquad (2)$$

if $$e_{-1} = \frac{1-(1-e_{-2}^2)^{\frac{1}{2}}}{1+(1-e_{-2}^2)^{\frac{1}{2}}}, \qquad e = \frac{1-(1-e_{-1}^2)^{\frac{1}{2}}}{1+(1-e_{-1}^2)^{\frac{1}{2}}},$$

we should have a similar equation between s_{-2}, s_{-1}, and s; and generally we may obtain a series of equations

$$\frac{e_n x_n + s_n - (1+e_n)s_{n-1}}{1-e_n^2} = \frac{1+e_{n+1}}{2} \cdot \frac{e_{n+1} x_{n+1} + s_{n+1} - (1+e_{n+1})s_n}{1-e_{n+1}^2}$$

from which, any two consecutive values of s being known, any others in the series may be determined.

By this process the rectification of any proposed ellipse may be made to depend on that of another ellipse of either much greater or much less eccentricity: in either case an approximation may be made by a series, as in (Art. 17.)

(22.) The arc of an hyperbola may be thus referred to elliptic arcs: let 1 be the eccentricity, e the semi-axis major, and S the arc; then the equation of the hyperbola is

$$y = \frac{b}{e}(x^2 - e^2)^{\frac{1}{2}}.$$

Let $x = e\{1 + (1-e^2)(\tan\phi)^2\}^{\frac{1}{2}}$, then $y = (1-e^2)\tan\phi$;

and $$S = \tan\phi\{1-(e\sin\phi)^2\}^{\frac{1}{2}} - e^2 \int_\phi \frac{(\cos\phi)^2}{\{1-(e\sin\phi)^2\}^{\frac{1}{2}}};$$

$$= \tan\phi\{1-(e\sin\phi)^2\}^{\frac{1}{2}} - s + (1-e^2)(s - e . d_e s).$$

By changing e, ϕ, s, S, into e_1, ϕ_1, s_1, S_1, and substituting for $s_1 - e_1 . d_{e_1} s_1$ its value obtained from (1), we obtain

$$S_1 = \tan\phi_1\{1-(e_1\sin\phi_1)^2\}^{\frac{1}{2}} + 2e_1\sin\phi_1 + s_1 - 2(1+e_1)s.$$

(23.) $\displaystyle\int_x \frac{X}{\{a + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4\}^{\frac{1}{2}}}$. which depends on the forms $\int_u \frac{1}{Q}, \int_u \frac{u^2}{Q}, \int_u \frac{1}{(u^2+a)Q},$

where $Q=\{(1\pm p^2u^2)(1\pm q^2u^2)\}^{\frac{1}{2}}$, may be reduced to the form

$$\int_\varphi \frac{A_1+A_2(\sin\phi)^2}{A_3+A_4(\sin\phi)^2}\cdot\frac{1}{\{1-(e\sin\phi)^2\}^{\frac{1}{2}}},$$

by assuming $pu=\tan\phi$, and $1-\frac{q^2}{p^2}=e^2$, when p and q are positive;

$pu=\sin\phi, \ldots \frac{q^2}{p^2}=e^2$, when p and q are negative:

the value of the two first, and sometimes of the last, may be represented by elliptic arcs.

(*L. C. D.* 502—11; *Legendre, Exerc. du Calc. Int.*)

QUADRATURE.

(24.) Let $y=\phi(x)$ be the equation of a plane curve, and A the area contained between the axis of x, the curve, and any two ordinates $\phi(a)$, and $\phi(b)$, then

$$d_xA=\phi(x); \quad \text{and } A=\left(\int_{x=b}-\int_{x=a}\right)\phi(x).$$

$$\int_x y=xy-\int_y x=y\phi^{-1}(y)-\int_y\phi^{-1}(y);$$

this will frequently be found a convenient transformation, when the function ϕ is not readily integrable.

(25.) Let $u=\phi(\theta)$ be the polar equation of a curve, and A the area contained between two radii and the curve, then

$$d_\theta A=\tfrac{1}{2}\overline{\phi(\theta)}|^2, \qquad \text{and } A=\tfrac{1}{2}\int_\theta\overline{\phi(\theta)}|^2.$$

The value of A between the limits $\theta=\alpha$, and $\theta=\beta$, is

$$\left(\int_{\theta=\beta}-\int_{\theta=\alpha}\right)\overline{\phi(\theta)}|^2.$$

(26.) Let $z=\phi(x,y)$ be the equation of a surface, and S the area of the surface intercepted between the planes xz, yz, and two planes respectively parallel to them, then

$$d_xd_yS=\{1+(d_xz)^2+(d_yz)^2\}^{\frac{1}{2}},$$

and $$S=\int_x\int_y\{1+(d_xz)^2+(d_yz)^2\}^{\frac{1}{2}}.$$

If the surface is bounded by the plane xz, a plane parallel to yz, and the plane $y=ux$, then

$$d_x d_u S = x\{1+(d_x z)^2+(d_y z)^2\}^{\frac{1}{2}},$$

and

$$S = \int_u \int_x x\{1+(d_x z)^2+(d_y z)^2\}^{\frac{1}{2}}.$$

It is indifferent in what order the integration is performed, but the proposed limits must be assigned after each integration.

CUBATURE.

(27.) Let V be the volume contained between the three co-ordinate planes, two planes parallel to xz, and yz, and the portion of the surface intercepted between them; and let $z=\phi(x,y)$ be the equation of the surface, then

$$d_x d_y V=\phi(x,y), \qquad \text{and } V=\int_x \int_y \phi(x,y).$$

If the volume is contained between the co-ordinate planes xy, xz, a plane parallel to yz, the plane $y=ux$, and the portion of the surface $z=\phi(x,y)$ intercepted between them, then

$$d_u d_x V = x.\phi(x,y), \qquad \text{and } V=\int_u \int_x x.\phi(x,y).$$

If the surface is generated by the revolution of a plane curve round the axis of x, then

$$V=\pi \int_x y^2.$$

TRANSFORMATION OF CO-ORDINATES.

(28.) Let x and y be functions of t and u, the co-ordinates of another system, such that

$$d_t x = P_1 + Q_1 d_t u, \qquad d_t y = P_2 + Q_2 d_t u;$$

then $\int_x \int_y V$ becomes $\int_t \int_u V(P_1 Q_2 - P_2 P_1)$, the values of x and y in terms of t and u being substituted in V.

Let x, y, and z, be given functions of a new system of co-ordinates t, u, v, such that

$$d_t x = P_1 + Q_1 d_t u + R_1 d_t v,$$
$$d_t y = P_2 + Q_2 d_t u + R_2 d_t v,$$
$$d_t z = P_3 + Q_3 d_t u + R_3 d_t v;$$

then $\int_x\int_y\int_z V$ becomes

$$\int_t\int_u\int_v V\{P_1(Q_2R_3-Q_3R_2)-Q_1(P_2R_3-P_3R_2)+R_1(P_2Q_3-P_3Q_2)\}$$

If the equation of a surface be transformed from rectilinear to polar co-ordinates, then

$$\int_x\int_y\int_z V \text{ becomes } \int_\phi\int_\psi\int_r Vr^2\cos\phi.$$

(*L. C. D.* 531; *Lagr. Berl. Mem. an.* 1773.)

(29.) *Conditions which render a curve quadrable.*

Let $\int_x y=X$, and let X be resolved into $X_1+\int_x X_2$, in which X_1 is an algebraic function of x, and $\int_x X_2$ a function involving transcendants, which however vanishes when $x=a$, $x=b$, &c. and let

$$\int_x X_2=\int_x\phi(x)-\int_z\phi(z),$$

in which $z=x+X_3(x-a)(x-b)$ &c. X_3 not containing any factors of the first degree, then if

$$y=d_x X_1+\phi(x)-d_x z.\phi(z),$$

the curve is quradrable the limits $x=a$, $x=b$, &c.

(30.) *Conditions which render a curve rectifiable.*

Let $d_x y=p$, then

$$y=px-\int_p x, \text{ and } s=x(1+p^2)^{\frac{1}{2}}-\int_p\frac{px}{(1+p^2)^{\frac{1}{2}}}; \qquad (1)$$

assume $\int_p x=P$, and $\int_p\frac{px}{(1+p^2)^{\frac{1}{2}}}=Q$, then $p=\frac{d_pQ}{\{1+(d_pQ)^2\}^{\frac{1}{2}}}$: by eliminating x between the equations (1), the ordinate of a rectifiable curve will be obtained from the first, and the expression for the arc of that curve, from the second.

(*L. C. D.* 735—7.)

TRAJECTORIES.

(31.) Let $\phi(x,y,a)=0$ be the equation of a system of curves, of which a is the variable parameter, and let the derived equation be

$$M+Nd_xy=0; \qquad (1)$$

then by eliminating a between these equations, we obtain

$$\psi(d_xy,x,y)=0.$$

Let u be the value of $d_x y$ obtained from this equation, and θ the angle at which the trajectory is required to intersect the given curves, then the differential equation of the trajectory is

$$(1 + u \tan\theta) d_x y - u + \tan\theta = 0. \qquad (2)$$

If v is the value of $d_x y$ obtained from (1), then the equation of the trajectory may be found by eliminating a between the equations

$$\phi(x,y,a) = 0,$$

$$(1 + v \tan\theta) d_x y - v + \tan\theta = 0. \qquad (3)$$

If the orthogonal trajectory is required, (2) becomes

$$u d_x y + 1 = 0.$$

If the elimination of a presents any difficulties, and one of the variables may be readily separated in the proposed equation, then a being considered variable, we may obtain from (3) the equation $$(1 + v^2) \tan\theta . d_a x - (1 - v \tan\theta) d_a y = 0, \qquad (4)$$ from which the relation between a and x may be determined, by means of which the required trajectory may be described.

If only $d_x y = v$, the differential equation of the proposed system of curves is given, then (4) becomes

$$(1 + v^2) \tan\theta . d_a x - (1 - v \tan\theta) \textstyle\int_x d_a v = 0,$$

which is an equation of the first order, if $d_a v$ is integrable with respect to x. (*L. D. C.* 402—5; *L. C. D.* 681—3.)

REMARKABLE ALGEBRAIC CURVES.

(32.) *The conchoid of Nicomedes.* Let the pole be in the axis of x, at a distance b from the origin, then the equation between rectangular co-ordinates is

$$x^2 y^2 = (b + x)^2 . (a^2 - x^2).$$

The equation of the tangent is

$$y_1 = \frac{(x^3 + ab^2) x_1 - 2ab^2 x - b^2 x^2 + a x^3}{x^2 (b^2 - x^2)^{\frac{1}{2}}}.$$

The points of inflexion may be determined from the equation $$x^3 + 3bx^2 - 2a^2 b = 0.$$

The polar equation is $u = \frac{b}{\cos\theta} \pm a$.

If $a < b$, there are four points of inflexion, and the pole is a conjugate point: if $a = b$, there is a cusp, and if $a > b$, a node, at the pole, and the curve has only two points of inflexion. The axis of y is an asymptote to the curve.

(33.) *The cissoid of Diocles.* The equation between rectangular co-ordinates is $$y^2 = \frac{x^3}{a-x}.$$

The equation of the tangent is

$$y_1 = \frac{x^{\frac{1}{2}}}{2(a-x)^{\frac{3}{2}}}\{(3a-2x)x_1 - ax\}.$$

The subtangent $= \frac{2x(a-x)}{3a-2x}$.

The curve has a cusp at the origin, and two hyperbolic branches, to which the ordinate corresponding to $x = a$, is an asymptote.

The polar equation is $u = \frac{2a(\sin\theta)^2}{\cos\theta}$.

The area contained by the curve and the asymptote $= \frac{3}{4}\pi a^2$. (*Wood*, *Alg.* 496, 7.)

(34.) *The Witch:* $y = \frac{a}{x}(ax - x^2)^{\frac{1}{2}}$.

The equation of the tangent is

$$y_1 = -\frac{a}{2x(ax-x^2)^{\frac{1}{2}}}(ax_1 + 2x^2 - 3ax).$$

The subtangent $= -\frac{2(ax-x^2)}{a}$.

The axis of y is an asymptote to the curve.

(*Agnesi*, *Analyt. Inst.* V. I. Art. 242.)

(35). *The Lemniscata.* The equation between rectangular co-ordinates is $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

The polar equation is $u = a.\cos 2\theta$.

The area of the curve $= a^2$.

(36.) *The semicubical parabola.* The equation is

$$ay^2 = x^3.$$

This curve is the evolute of the common parabola: the length of the arc measured from the origin is

$$\frac{8a}{27}\left\{\left(1+\frac{9x}{4a}\right)^{\frac{3}{2}}-1\right\}.$$

(37.) The equation of the evolute of the ellipse or hyperbola

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1,$$

is $$\left(\frac{x_1}{a_1}\right)^{\frac{2}{3}} \pm \left(\frac{y_1}{b_1}\right)^{\frac{2}{3}} = 1;$$

in which $$a_1 = \frac{a^2 \mp b^2}{a}, \quad \text{and } b_1 = \frac{a^2 \mp b^2}{b}.$$

(38.) *The Trisectrix.* The equation is

$$(x^2 - 2ax + y^2)^2 = a^2(x^2 + y^2).$$

The polar equation is $u = a(2\cos\theta \pm 1)$.

(*Peacock, Ex.* pp. 160, 70, 73.)

Remarkable Transcendental Curves.

(39.) *The quadratrix of Dinostratus.*

$$y = (1 - x)\tan \tfrac{1}{2}\pi x.$$

The polar equation is $u = \dfrac{2\theta}{\pi \sin\theta}$.

The quadratrix of Tschirnhausen.

$$y = \sin \tfrac{1}{2}\pi x.$$

The polar equation is $u\sin\theta = \cos(\tfrac{1}{2}\pi u \cos\theta)$.

(40.) *The logarithmic curve.*

$$y = a^x.$$

The subtangent $= \dfrac{1}{\log_e a}$.

The axis of $-y$ is an asymptote to the curve.

(41.) *The cycloid, and cycloidal curves.*

Let the vertex of the curve be the origin, and its diameter the axis of x, then the general form of the equations of a cycloidal curve is

$$x = a.\text{vers}\,\theta, \qquad y = a(n\sin\theta + m\theta).$$

The equations of a trochoidal curve are

$$x = a\,\text{vers}\,\theta, \qquad y = a(\sin\theta + m\theta).$$

The equations of the common cycloid are

$$x = a\,\text{vers}\,\theta, \qquad y = a(\sin\theta + \theta).$$

The length of the arc is four times the diameter of the generating circle. The area is three times the area of the generating circle.

The radius of curvature $= 4a.\cos\frac{1}{2}\theta$.

The evolute of each semi-cycloid is an equal semi-cycloid on a parallel base, the vertex of which coincides with the cusp of the former.

The maximum ordinate of the *curtate* cycloid corresponds to

$$x = a(1+m).$$

The *prolate* cycloid has a point of inflexion corresponding to

$$x = \frac{a}{m}(1+m).$$

The whole trochoidal area $= 2m + 1$ times the area of the generating circle.

The companion of the cycloid:

$$x = a\,\text{vers}\,\theta, \qquad y = a\theta.$$

The whole area equals twice the area of the generating circle.

The equations of the *harmonic curve* are

$$x = \frac{a}{m}\,\text{vers}\,\theta, \qquad y = a\theta.$$

The radius of curvature $= \dfrac{a\{m^2 + (\sin\theta)^2\}^{\frac{3}{2}}}{m^2.\cos\theta}$.

(*Peacock*, *Ex.* p. 186.)

(42.) *Epicycloidal curves.*

Let a be the radius of the base, b that of the generating circle, and let the centre of the base be the origin, and the radius passing through the first point of contact, the axis of x; θ the angle contained between the axis of x, and the radius passing through the point of contact in any new position of the generating circle; then the equations of the *epicycloid* are

$$x = (a+b)\cos\theta - b\cos\frac{a+b}{b}\theta,$$

$$y = (a+b)\sin\theta - b\sin\frac{a+b}{b}\theta:$$

those of the corresponding *hypocycloid*, in which b is negative; are

$$x = (a-b)\cos\theta + b\cos\frac{a-b}{b}\theta,$$

$$y = (a-b)\sin\theta - b\sin\frac{a-b}{b}\theta.$$

If b_1 is the distance of the generating point from the centre of the generating circle, then the equations of the *epitrochoid* are

$$x = (a+b)\cos\theta - b_1\cos\frac{a+b}{b}\theta,$$

$$y = (a+b)\sin\theta - b_1\sin\frac{a+b}{b}\theta:$$

and those of the corresponding *hypotrochoid* are

$$x = (a-b)\cos\theta + b_1\cos\frac{a-b}{b}\theta,$$

$$y = (a-b)\sin\theta - b_1\sin\frac{a-b}{b}\theta.$$

The angle contained between the tangent, and the axis

$$\text{in the epicycloid} = \frac{a+2b}{2b}.\theta,$$

$$\ldots\ldots \text{hypocycloid} = \pi - \frac{a-2b}{2b}\theta.$$

$$\text{The arc of the epicycloid} = \frac{4b}{a}(a+b)\left(1-\cos\frac{a\theta}{2b}\right),$$

$$= \frac{4b}{a}(a+b)\,\text{vers}\,\tfrac{1}{2}\phi;$$

$$\ldots\ldots\ldots\ \text{hypocycloid} = \frac{4b}{a}(a-b)\,\text{vers}\,\tfrac{1}{2}\phi.$$

The radius of curvature of the epicycloid $= -\dfrac{4b(a+b)}{a+2b}\sin\tfrac{1}{2}\phi$: the evolute is an epicycloid similar to the original curve, the radii of the base, and generating circle being respectively

$$\frac{a^2}{a+2b},\quad \frac{ab}{a+2b}.$$

The equations between p and u, the centre of the base being the pole;

$$\text{in the epicycloid, } p = e\left(\frac{u^2-a^2}{e^2-a^2}\right)^{\frac{1}{2}}, \text{ where } e = a+2b,$$

$$\ldots\ \text{hypocycloid, } p = e\left(\frac{a^2-u^2}{a^2-e^2}\right)^{\frac{1}{2}},\ \ldots\ e = a-2b.$$

The area contained by the axis, the radius vector, and the curve

$$= \frac{1}{2}(a+b)(a+2b)\left(\theta-\frac{b}{a}\sin\phi\right), \text{ in the epicycloid,}$$

$$= \frac{1}{2}(a-b)(a-2b)\left(\theta-\frac{b}{a}\sin\phi\right),\ \ldots\ \text{hypocycloid.}$$

If $a : b$ is a finite ratio, the curve may be represented by a finite algebraical equation. The equation of the *cardioide*, in which $a=b$, is

$$\overline{y^2+x(x-a)}\,|^2 = a^2\{y^2+(x-a)^2\}.$$

If the diameter of the curve is the axis of x, and the cusp the origin, the polar equation is

$$u = a(1-\cos\psi).$$

The equation of the epitrochoid in which $a=b$, the origin being at the first point of contact, is

$$x^4+2b_1x^3-(a^2-b_1^2)x^2+2(x+b_1)xy^2-a^2y^2+y^4=0.$$

The equation of the epicycloid with two cusps, in which $b=\frac{1}{2}a$, is $$a^4y^2=(x^2+y^2-a^2)^3.$$

The equations of the hypotrochoid, in which $b=\frac{1}{2}a$, are

$$x=(\tfrac{1}{2}a+b_1)\cos\theta,$$
$$y=(\tfrac{1}{2}a-b_1)\sin\theta,$$

the equations of an ellipse, of which the semi-axes are $\frac{1}{2}a+b_1$, and $\frac{1}{2}a-b_1$. (*Peacock*, *Ex.* p. 192.)

(43.) *Spirals.* The equation of the spiral of Archimedes, is

$$u=a\theta;$$

the equation between p and u; $p=\dfrac{u^2}{(a^2+u^2)^{\frac{1}{2}}}$.

The equation of the *Lituus* is $u^2=\dfrac{a^2}{\theta}$.

The first radius produced is an asymptote to the curve, and there is a point of contrary flexure.

(44.) *Cotes's Spirals.*

[1] *The hyperbolic spiral*; $u=\dfrac{a}{\theta}$.

The equation between p and u; $p=\dfrac{au}{(a^2+u^2)^{\frac{1}{2}}}$.

The subtangent $=a$.

The area $=\frac{1}{2}au$.

The length of the arc $=a\log_e\dfrac{u}{(a^2+u^2)^{\frac{1}{2}}+a}+(a^2+u^2)^{\frac{1}{2}}-a$.

[2] $p=\dfrac{bu}{(a^2-u^2)^{\frac{1}{2}}}$, or $\theta=\dfrac{b}{c}\log_e\dfrac{c+(c^2-u^2)^{\frac{1}{2}}}{u}$,

where $c=(a^2-b^2)^{\frac{1}{2}}$.

[3] $p = \frac{bu}{(a^2+u^2)^{\frac{1}{2}}}$, and $a < b$; then the polar equation is

$$\theta = \frac{b}{c} \log_e \frac{(u^2+c^2)^{\frac{1}{2}} - c}{u},$$

where $c = (a^2 - b^2)^{\frac{1}{2}}$.

[4] $p = \frac{bu}{(a^2+u^2)^{\frac{1}{2}}}$, and $a > b$; then

$$u = c \cdot \sec \frac{c}{b} \theta,$$

where $c = (b^2 - a^2)^{\frac{1}{2}}$.

[5] *The logarithmic spiral:* $p = \frac{b}{a} u$, and

$$\theta = \frac{b}{(a^2 - b^2)^{\frac{1}{2}}} \log_e \frac{u}{a}.$$

The angle contained between the radius vector and the curve is constant, and $= \tan^{-1} \frac{b}{(a^2 - b^2)^{\frac{1}{2}}}$.

The evolute is a similar spiral.

(45.) The polar equation of the *involute of the circle*, is

$$\frac{(u^2 - a^2)^{\frac{1}{2}}}{a} = \theta + \sec^{-1} \frac{u}{a}.$$

(46.) *The tractory:* The differential equation is

$$d_x y = - \frac{y}{(a^2 - y^2)^{\frac{1}{2}}};$$

whence $$x = a \log_e \left\{ \frac{a + (a^2 - y^2)^{\frac{1}{2}}}{y} \right\} - (a^2 - y^2)^{\frac{1}{2}}.$$

The evolute of this curve is the *catenary*.

The equation of the syntractory is

$$x = a \log_e \frac{b + (b^2 - y^2)^{\frac{1}{2}}}{y} - (b^2 - y^2)^{\frac{1}{2}}.$$

If $b < a$, this curve has a point of inflexion, corresponding to

$$y = \frac{b \,.\, a^{\frac{1}{2}}}{(2a + b)^{\frac{1}{2}}}.$$

If $b > a$, there is no point of inflexion, but when $y = (ab)^{\frac{1}{2}}$, the tangent is perpendicular to the axis.

If $b < 0$, there is a point of inflexion, when

$$y = \frac{-b \,.\, a^{\frac{1}{2}}}{(2a + b)^{\frac{1}{2}}}. \quad \textit{(Peacock, Ex. p. 174.)}$$

(47.) *The catenary.* Let the curve be referred to its diameter, and a tangent at the vertex, and let s be the length of the arc; the differential equation is

$$d_x s = \frac{(c^2 + s^2)^{\frac{1}{2}}}{s},$$

whence $x + c = (s^2 + c^2)^{\frac{1}{2}}$, or $s = (x^2 + 2cx)^{\frac{1}{2}}$.

The equations between y and s are

$$\frac{y}{c} = \log_\epsilon \frac{s + (s^2 + c^2)^{\frac{1}{2}}}{c}, \qquad s = \tfrac{1}{2} c \left(\epsilon^{\frac{y}{c}} - \epsilon^{-\frac{y}{c}}\right),$$

the equations between x and y are

$$x + c = \tfrac{1}{2} c \left(\epsilon^{\frac{y}{c}} + \epsilon^{-\frac{y}{c}}\right),$$

$$\frac{y}{c} = \log_\epsilon \frac{x + c + (x^2 + 2cx)^{\frac{1}{2}}}{c}.$$

(*Whewell, Mech.* 111—3.)

(48.) *The elastic curve.* Let the extremities of the elastic line be situated in the axis of y, at equal distances from the origin, and let α be the angle at which the curve passes through the origin; $b^2 = -a^2 \sin \alpha$, and $c^2 = a^2 - b^2$, then the differential equations of the curve are

$$d_x y = \frac{a^2 - c^2 + x^2}{\overline{(c^2 - x^2)(2a^2 - c^2 + x^2)}|^{\frac{1}{2}}},$$

$$d_x s = \frac{a^2}{\overline{(c^2 - x^2)(2a^2 - c^2 + x^2)}|^{\frac{1}{2}}}.$$

Species [1]. Let $\frac{c}{a}$ be very small: then, neglecting c^2 and x^2, we have

$$d_xy = \frac{a}{\{2(c^2-x^2)\}^{\frac{1}{2}}}, \text{ whence } x = c.\sin\frac{y\sqrt{2}}{a},$$

$$s = y + \frac{c^2}{2a^2}y + \frac{c^2}{4\sqrt{2}.a}\sin\frac{2\sqrt{2}.y}{a}.$$

[2]. $\frac{c}{a} < 1$. In this case the values of y and s can only be obtained by series.

[3] $\frac{c}{a} = 1$. In this case $d_xy = \frac{x^2}{(a^4-x^4)^{\frac{1}{2}}}$, and

$$d_xs = \frac{a^2}{(a^4-x^4)^{\frac{1}{2}}}.$$

Let l be the length of the curve, and h the distance of the extremity of the curve from the origin, then

$$l = \frac{\pi a}{\sqrt{2}}\left\{1 + \frac{1^2}{2^2}.\frac{1}{2} + \frac{1^2.3^2}{2^2.4^2}.\frac{1}{4} + \&c.\right\},$$

$$h = \frac{\pi a}{\sqrt{2}}\left\{1 + \frac{1^2}{2^2}.\frac{3}{1}.\frac{1}{2} + \frac{1^2.3^2}{2^2.4^2}.\frac{5}{3}.\frac{1}{4} + \&c.\right\},$$

$$lh = \pi a^2.$$

[4] $\frac{c}{a} > 1$, and $h > 0$. In this case there is a maximum ordinate when $x^2 = a^2.\sin a$.

[5] $\frac{c}{a} > 1$, and $h = 0$. In this case the curve returns into itself.

[6] $\frac{c}{a} > 1$, and $h < 0$. The branches of the curve cross each other.

[7] $\frac{c}{a} = \sqrt{2}$. The axis of y is an asymptote, and the equation of the curve is

$$y = -(c^2 - x^2)^{\frac{1}{2}} + \tfrac{1}{2}c \log_e \frac{c + (c^2 - x^2)^{\frac{1}{2}}}{x}.$$

[8] $\dfrac{c}{a} > \surd 2$. Let $c^2 = 2a^2 + g^2$, then

$$d_x y = \frac{x^2 - \frac{1}{2}c^2 - \frac{1}{2}g^2}{\overline{(c^2 - x^2)(x^2 - g^2)}\big|^{\frac{1}{2}}}.$$

g is the minimum, and c the maximum, value of x: beyond these limits x is impossible.

y is a maximum, when $x = \overline{\tfrac{1}{2}(c^2 + g^2)}\big|^{\frac{1}{2}}$.

[9] $\dfrac{c}{a} = \infty$. In this case the curve is a circle.

(*Whewell, Mech. Appendix.*)

CALCULUS OF VARIATIONS.

For a history of the origin and principles of the calculus of variations, see (*Woodhouse's Isoper. Prob.*; *L. C. D.* 825—43.)

(1.) $$\delta d_x u = d_x \delta u; \qquad \delta \int_x u = \int_x \delta u.$$
$$\delta d_x^n u = d_x^n \delta u; \qquad \delta \int_x^n u = \int_x^n \delta u.$$

If $u = \phi(x, y, y', y'', \text{\&c.})$, then

$$\delta u = d_x u \delta x + d_y u \delta y + d_{y'} u \delta y' + \text{\&c.}$$
$$\delta y' = d_x(\delta y - y' \delta x) + y''.\delta x,$$
$$\text{\&c.} = \text{\&c.}$$
$$\delta d_x^n y = d_x^n(\delta y - y' \delta x) + d_x^{n+1} y.\delta x;$$

from which δu may be obtained in terms of δy and δx.

(2.) $$\delta \int_x \phi(x, y, y', y'', \text{\&c.}) = \delta \int_x V$$
$$\left.\begin{aligned} = V\delta x &+ \{d_{y'} V - d_x d_{y''} V + \text{\&c.}\}(\delta y - y' \delta x) \\ &+ \{d_{y''} V - \text{\&c.}\}\, d_x(\delta y - y' \delta x) \\ &+ \text{\&c.} \end{aligned}\right\} \quad (1)$$
$$+ \int_x \{d_y V - d_x d_{y'} V + d_x^2 d_{y''} V - \text{\&c.}\}(\delta y - y' \delta x) \quad (2)$$

In order that V may be integrable with respect to x, we must have $$d_y V - d_x d_{y'} V + d_x^2 d_{y''} V - \text{\&c.} = 0.$$

From this last equation it follows that the quantities

$$\int_x d_y V,$$
$$\int \{\int_x d_y V - d_{y'} V\},$$
$$\int_x \{\int_x (\int_x d_y V - d_{y'} V) + d_{y''} V\}, \text{ \&c.}$$

may always be determined.

(*L. D. C.* 456—67; *L. C. D.* 844—53.)

(3.) If $V=\phi\{x,y,y',y'',\ \&c.\int_x\psi(x,y,y',\ \&c.)\}$

$=\phi(x,y,y',y'',\ \&c.\int_x V_1)$,

let $\delta y-y'\delta x=\omega$, $\int_x V_1=u$, and $\int_x d_u V=U$, then

$$\begin{aligned}
\delta\int_x V = V\delta x &+ \{d_{y'}V-d_x d_{y''}V+d_x^2 d_{y'''}V-\&c.\}\,\omega \\
&+ \{d_{y''}V-d_x d_{y'''}V+\&c.\}\,d_x\omega \\
&+ \{d_{y'''}V-\&c.\}\,d_x^2\omega \\
&+ \&c. \\
&+ \int_x\{d_y V-d_x d_{y'}V+d_x^2 d_{y''}V-\&c.\}\,\omega \\
&+U\left\{\begin{aligned}
&\{d_{y'}V_1-d_x d_{y''}V_1+d_x^2 d_{y'''}V_1-\&c.\}\,\omega \\
&+\{d_{y''}V_1-d_x d_{y'''}V_1+\&c.\}\,d_x\omega \\
&+\{d_{y'''}V_1-\&c.\}\,d_x^2\omega \\
&+\ \&c. \\
&+\int_x\{d_y V_1-d_x d_{y'}V_1+d_x^2 d_{y''}V_1-\&c\}\,\omega
\end{aligned}\right. \\
&- \{U d_{y'}V_1-d_x U d_{y''}V_1+\&c\}\,\omega \\
&- \{U d_{y''}V_1-d_x U.d_{y''}V_1+\&c.\}\,d_x\omega \\
&- \{U d_{y''}V_1-\&c.\}\,d_x^2\omega \\
&- \&c. \\
&- \int_x\{U d_y V_1-d_x U d_{y'}V_1+d_x^2 U d_{y''}V_1-\&c.\}\,\omega.
\end{aligned}$$

(*L. C. D.* 854.)

(4.) Let $V=\phi\{x,y,z,\ p,\ q,\ r,\ s,\ t,\ldots\&c.\}$,

In which $p,\ q,\ r,\ s,\ t,\ u,\ v,\ w,\ o,$ represent $d_x z,\ d_y z,\ d_x^2 z,\ d_x d_y z,\ d_y^2 z,\ d_x^3 z,\ d_x^2 d_y z,\ d_x d_y^2 z,\ d_y^3 z,$ respectively, and let $\omega=\delta z-p\delta x-q\delta y$, then

$$\begin{aligned}
\delta\int_x\int_y V = \omega &\left\{\begin{aligned}&d_s V-d_x d_v V+\&c.\\ &\qquad -d_y d_w V+\&c.\end{aligned}\right\} \\
&+ d_x\omega\,\{d_v V-\&c.\}+\&c. \\
&+ d_y\omega\,\{d_w V-\&c.\}+\&c. \\
&+ \int_y V\delta x+\int_x V\delta y
\end{aligned}$$

$$\left.\begin{array}{r} +\int_x\omega\{d_q V-d_y d_t V \;+\; d_y^2 d_s V-\&c. \\ -d_x d_s V+d_x d_y d_w V-\&c. \\ +d_x^2 d_v V-\&c. \end{array}\right\}$$

$$\left.\begin{array}{r} +\int_y\omega\{d_p V-d_x d_r V \;+\; d_x^2 d_u V-\&c. \\ -d_y d_s V+d_x d_y d_v V-\&c. \\ +d_y^2 d_w V-\&c. \end{array}\right\}$$

$$\left.\begin{array}{r} +\int_x d_y\omega\{d_t V-d_y d_o V+\&c. \\ -d_x d_w V+\&c. \\ +\&c. \end{array}\right\}$$

$$\left.\begin{array}{r} +\int_y d_x\omega\{d_r V-d_x d_u V+\&c. \\ -d_y d_v V+\&c. \\ +\&c. \end{array}\right\}$$

$$+\int_x d_y^2\omega\{d_o V-\&c.\}+\&c.$$

$$+\int_y d_x^2\omega\{d_u V-\&c.\}+\&c.$$

$$\left.\begin{array}{r} +\int_x\int_y\omega\{d_z V-d_x d_p V \;+\; d_x^2 d_r V-\&c. \\ -d_y d_q V+d_x d_y d_s V-\&c. \\ +d_y^2 d_t V-\&c. \\ -\&c. \end{array}\right\}$$

(*L. C. D.* 862.)

MAXIMA AND MINIMA OF INDETERMINATE INTEGRALS.

(5.) Since the part (1) of the value of $\delta\int_x V$ has been obtained by integration, it must be taken between limits: let the values of V, x, y, y'; &c. corresponding to the first limit be

$$V_1,\ x_1,\ y_1,\ y'_1;\ \&c.$$

and to the second v_2, x_2, y_2, y'_2; &c. then

$$\begin{aligned} \delta\int_x V = {} & V_2\delta x_2-V_1\delta x_1 \\ & +\{d_{y'_2}V_2-d_{x_2}d_{y''_2}V_2+\&c.\}(\delta y_2-y'_2\delta x_2) \\ & -\{d_{y'_1}V_1-d_{x_1}d_{y''_1}V_1+\&c.\}(\delta y_1-y'_1\delta x_1) \qquad (3) \\ & +\{d_{y''_2}V_2-\&c.\}d_{x_2}(\delta y_2-y'_2\delta x_2) \end{aligned}$$

$$
\begin{aligned}
&-\{d_{y'_1} V_1 - \&\text{c.}\} . d_{x_1}(\delta y_1 - y'_1 \delta x_1)\\
&+ \ \&\text{c.} \quad - \qquad \&\text{c.}\\
&+\textstyle\int_x \{d_y V - d_x d_{y'} V + d_x^2 d_{y''} V - \&\text{c.}\} (\delta y - y' \delta x). \qquad (2)
\end{aligned}
$$

(6.) If a maximum or minimum value of $\int_x V$ be required, then $\delta \int_x V = 0$, and hence the quantities (2), (3) must be separately equal to nothing. Eliminating therefore between (3) and any given conditional equations as many as possible of the quantities δx_1, δx_2, δy_1, δy_2, &c., and making the coefficients of the rest severally $=0$, we may obtain the value required.

The required result may frequently be more easily obtained from the condition

$$d_y V - d_x d_{y'} V + d_x^2 d_{y''} V - \&\text{c.} = 0.$$

(7.) If in addition to the condition $U_1 =$ *maximum* or *minimum*, we have also $U_2 = 0$, then $\delta U_2 = 0$. Both these conditions may be expressed by the equation

$$\delta U_1 + a \delta U_2 = 0,$$

a being a constant to be determined by the nature of the question. From this equation δx_1, δx_2 &c. must be eliminated, as in the preceding case.

Similarly if the condition $U_3 = 0$ be also given, then

$$\delta U_1 + a \delta U_2 + b \delta U_3 = 0;$$

and so on for any number of independent conditional equations.

(*Airy*, *Math. Tracts.*)

CALCULUS OF FINITE DIFFERENCES.

DIRECT METHOD OF DIFFERENCES.

(1.) $\Delta_x({}^1u_x+{}^2u_x+\&c.+{}^nu_x+const.)=\Delta_x{}^1u_x+\&c.+\Delta_x{}^nu_x.$ [*]

$\Delta_x a.u_x=a\Delta_x u_x,$ if a is either independent of x, or such a function of x, that its value is not changed, when x becomes $x+1$.

$$\Delta_x(u_x.v_x)=u_x\Delta_x v_x+v_{x+1}\Delta_x u_x.$$

$$\Delta_x\frac{u_x}{v_x}=\frac{v_x\Delta_x u_x-u_x\Delta_x v_x}{v_x.v_{x+1}}.$$

$$\Delta_x(u_x.u_{x+1}\ldots u_{x+n})=(u_{x+n+1}-u_x)u_{x+1}\ldots u_{x+n}.$$

$$\Delta_x\frac{1}{u_x.u_{x+1}\ldots u_{x+n}}=-\frac{u_{x+n+1}-u_x}{u_x.u_{x+1}\ldots u_{x+n+1}}.$$

(*L. D. C.* 497, 8; *Tr. L.* App. 342, 3.)

(2.) *Successive differences.*

$$\Delta_x(a_nx^n+a_{n-1}x^{n-1}+\&c.+a_1x+a)=na_nx^{n-1}+bx^{n-2}+\&c.$$

$$\Delta_x^2(a_nx^n+a_{n-1}x^{n-1}+\&c.+a_1x+a)=n(n-1)a_nx^{n-2}+\&c.$$

$$\&c. \qquad = \qquad \&c.$$

$$\Delta_x^n(a_nx^n+a_{n-1}x^{n-1}+\&c.+a_1x+a)=1.2.3\ldots n.a_n;$$

from which it appears that the n^{th} difference of an algebraic function of n dimensions is constant, and therefore that the differences of all orders superior to the n^{th} are equal to nothing.

---

[*] $\Delta_x\overset{n}{\mathrm{S}}_m{}^mu_x+const.=\overset{n}{\mathrm{S}}_m\Delta_x{}^mu_x.$

$$\Delta_x^2 u_x = u_{x+2} - 2u_{x+1} + u_x.$$

$$\Delta_x^3 u_x = u_{x+3} - 3u_{x+2} + 3u_{x+1} - u_x.$$

$$\&c. \quad = \quad \&c.$$

$$\Delta_x^n u_x = u_{x+n} - \frac{n}{1} u_{x+n-1} + \frac{n(n-1)}{1\ .\ 2} u_{x+n-2} - \&c. \quad {}^{\alpha}$$

$$u_{x+n} = u_x + \frac{n}{1}\Delta_x u_x + \frac{n(n-1)}{1\ .\ 2}\Delta_x^2 u_x + \&c. \quad {}^{\beta}$$

$$= (1+\Delta_x)^n u_x.$$

$$\Delta_x^n x^m = (x+n)^m - \frac{n}{1}(x+n-1)^m + \frac{n(n-1)}{1\ .\ 2}(x+n-2)^m - \&c. \quad {}^{\gamma}$$

$$(x+n)^n - \frac{n}{1}(x+n-1)^n + \frac{n(n-1)}{1\ .\ 2}(x+n-2)^n - \&c. = 1.2.3\ldots n.$$

$$\Delta^n 0^m = n^m - \frac{n}{1}(n-1)^m + \frac{n(n-1)}{1\ .\ 2}(n-2)^m + \&c. \quad {}^{\delta}$$

$$n^n - \frac{n}{1}(n-1)^n + \frac{n(n-1)}{1\ .\ 2}(n-2)^n - \&c. = 1.2.3\ldots n.$$

[α] $\Delta_x^n u_x = \overset{n+1}{S}_m (-1)^{m-1} \frac{\underline{|n}_{\,m-1}}{\underline{|m-1}} u_{x+n-m+1}.$

[β] $u_{x+n} = \overset{n+1}{S}_m \underline{|n}_{\,m-1} \ .\ \Delta_x^{m-1} : u_x.$

[γ] $\Delta_x^n x^m = \overset{n+1}{S}_r (-1)^{r-1} \frac{\underline{|n}_{\,r-1}}{\underline{|r-1}} (x+n-r+1)^m.$

[δ] $\Delta^n 0^m = \overset{n}{S}_r (-1)^{r-1} \frac{\underline{|n}_{\,r-1}}{\underline{|r-1}} (n-r+1)^m,$

or $\Delta^n : 0^m = \overset{n}{S}_r \frac{(-1)^{r-1}}{\underline{|r-1}} \ .\ \frac{(n-r+1)^m}{\underline{|n-r+1}}.$

(3) $\Delta_x {}^1u_x . {}^2u_x \ldots {}^mu_x = \{(1+{}^1\Delta)(1+{}^2\Delta)\ldots(1+{}^m\Delta)-1\} . {}^1u_x . {}^2u_x \ldots {}^mu_x.$ in which ${}^1\Delta$, &c. are merely used to denote that ${}^1u_x$, &c. are to be respectively annexed to them after the expansion. [α]

$\Delta_x^n {}^1u_x . {}^2u \ldots {}^mu_x = \{(1+{}^1\Delta_x)(1+{}^2\Delta_x)\ldots(1+{}^m\Delta_x)-1\}^n . {}^1u_x . {}^2u_x \ldots {}^mu_x.$

(*Tr. L.* App. 344—56; *L. D. C.* 499—507; *L. C. D.* 882—7.)

(4.) *Series involving the differences and differential coefficients of a function.*

$$\Delta_x u_x = \frac{1}{1} d_x u_x + \frac{1}{1.2} d_x^2 u_x + \frac{1}{1.2.3} d_x^3 u_x + \&c. \quad [β]$$

$$= (\epsilon^{d_x} - 1) u_x.$$

$$\Delta_x^n u_x = (\epsilon^{d_x} - 1)^n u_x. \quad [γ]$$

$$u_{x+n} = u_x + \frac{n}{1} d_x u_x + \frac{n^2}{1.2} d_x^2 u_x + \&c. \quad [δ]$$

$$= \epsilon^{n d_x} . u_x.$$

$$d_x u_x = \Delta_x u_x - \tfrac{1}{2}\Delta_x^2 u_x + \tfrac{1}{3}\Delta_x^3 u_x - \&c. \quad [ε]$$

$$= \log_\epsilon(1 + \Delta_x) u_x.$$

$$d_x^n u_x = \{\log_\epsilon(1 + \Delta_x)\}^n u_x. \quad [θ]$$

$$\Delta_x^n u_x = d_x^n u_x + \frac{\Delta^n 0^{n+1}}{1.2.3..(n+1)} d_x^{n+1} u_x + \&c. \quad [μ]$$

(*Tr. L.* App. 357—61; *L. C. D.* 929—38.)

[α] $\Delta_x \overset{m}{\mathrm{P}}_r {}^ru_x = \overset{m}{\mathrm{S}}_r \overset{\overline{m-r,r}}{\mathrm{C}}_{s,t} ({}^su_x . \Delta_x {}^tu_x)$: this notation will be explained in the Appendix.

[β] $\Delta_x u_x = \overset{\infty}{\mathrm{S}}_m d_x^m : u_x.$

[γ] $\Delta_x^n u_x = \overset{\infty}{\mathrm{S}}_{m_1} \overset{m_1}{\mathrm{S}}_{m_2} \ldots \overset{m_{n-1}}{\mathrm{S}}_{m_n} d_x^{m_1 - m_2 + 1} : d_x^{m_2 - m_3 + 1} : \ldots d_x^{m_{n-1} - m_n + 1} : d_x^{m_n} : u_x.$

[δ] $u_{x+n} = \overset{\infty}{\mathrm{S}}_m n^{m-1} . d_x^{m-1} : u_x.$

[ε] $d_x u_x = \overset{\infty}{\mathrm{S}}_m (-1)^{m-1} . \frac{\Delta_x^m u_x}{m}.$

[θ] $d_x^n u_x = \overset{\infty}{\mathrm{S}}_m \varpi^{m-1} a^n . \Delta_x^{n+m-1} u_x, \ \{a_{m-1} = \frac{(-1)^{m-1}}{m}\}.$

[μ] $\Delta_x^n u_x = \overset{\infty}{\mathrm{S}}_m \Delta^n 0^{n+m-1} . d_x^{n+m-1} : u_x.$

FUNCTIONS OF TWO OR MORE VARIABLES.

(5.) $\Delta_{x,y} u_{x,y} = \Delta_x u_{x,y} + \Delta_y u_{x,y} + \Delta_x \Delta_y u_{x,y}.$

$$= \{(1+\Delta_x)(1+\Delta_y) - 1\} u_{x,y}.$$

$$\Delta^n_{x,y} u_{x,y} = \{(1+\Delta_x)(1+\Delta_y) - 1\}^n u_{x,y}. \quad (L.C.D. \; 919, 20.)$$

$$\Delta^n{}_{x_1,x_2,\ldots x_m} u_{x_1,x_2\ldots x_m}$$

$$= \{(1+\Delta_{x_1})(1+\Delta_{x_2})\ldots(1+\Delta_{x_m}) - 1\}^n u_{x_1,x_2,\ldots x_m}.$$

$$\Delta^n_{x,y} u_{x,y} = u_{x+n,y+n} - n \,.\, u_{x+n-1,y+n-1} + \&c.$$

$$u_{x+n,y+n} = u_{x,y} + \frac{n}{1} \Delta_{x,y} u_{x,y} + \frac{n(n-1)}{1\,.\,2} \Delta^2_{x,y} u_{x,y} + \&c.$$

$$\Delta_{x,y} u_{x,y} = d_x u_{x,y} + d_y u_{x,y} + \frac{1}{1\,.\,2}(d_x^2 u_{x,y} + 2 d_x d_y u_{x,y} + d_y^2 u_{x,y}) + \&c.\text{[a]}$$

$$= (\epsilon^{d_x + d_y} - 1) u_{x,y}.$$

$$\Delta^n_{x,y} u_{x,y} = (\epsilon^{d_x + d_y} - 1)^n u_{x,y}.$$

$$\Delta^{m,n}_{x,\,y} u_{x,y} = (\epsilon^{d_x} - 1)^m . (\epsilon^{d_y} - 1)^n u_{x,y}.$$

$$\Delta_{x,y} u_{x,y} = \tfrac{1}{2}(\Delta_x u_{x,y} + \Delta_x u_{x,y+1}) + \tfrac{1}{2}(\Delta_y u_{x,y} + \Delta_y u_{x+1,y}).$$

(*Tr. L.* App. 362—6; *L. C. D.* 919, 20; 33, 4.)

INTERPOLATION OF SERIES.

(6.) Let u, u_1, u_2, &c. u_n, and v, v_1, v_2, &c. v_n, be corresponding known values of u and v, in the equation $u = \phi(v)$, in which the form of the function ϕ is unknown, and let it be required to determine the value of u_x, corresponding to any proposed value v_x, which is either between the limits v, v_n, or very near to one of them. Let v_r be the value of v preceding v_x, then if the intervals between the values of v are small, we may assume

---

[a] $\Delta^{-}_{x,y} u_{x,y} = \overset{\infty}{\mathrm{S}}_m \overset{m+1}{\mathrm{S}}_n \, d_x^{m-n+1} : d_y^{n-1} : u_{x,y}.$

$$u_x = a + a_1(v_x - v_r) + a_2(v_x - v_r)^2 + \&c. + a_n(v_x - v_r)^n,$$

which equation, when arranged according to powers of v_x, becomes $\quad u_x = b + b_1 v_x + b_2 v_x^2 + \&c. + b_n v_x^n$,

in which b, b_1, &c. remain to be determined. We have the $n+1$ equations

$$\begin{aligned} u &= b + b_1 v + b_2 v^2 + \&c. + b_n v^n, \\ u_1 &= b + b_1 v_1 + b_2 v_1^2 + \&c. + b_n v_1^n, \\ &\quad\&c. \qquad\qquad \&c. \\ u_n &= b + b_1 v_n + b_2 v_n^2 + \&c. + b_n v_n^n. \end{aligned}$$

$$\text{Let } \frac{u_1 - u}{v_1 - v} = A, \quad \frac{u_2 - u_1}{v_2 - v_1} = A_1, \ \&c. \ \frac{u_n - u_{n-1}}{v_n - v_{n-1}} = A_{n-1};$$

$$\frac{A_1 - A}{v_2 - v} = B, \quad \frac{A_2 - A_1}{v_3 - v_1} = B_1, \ \&c. \ \frac{A_{n-1} - A_{n-2}}{v_n - v_{n-2}} = B_{n-2};$$

$$\frac{B_1 - B}{v_3 - v} = C, \quad \frac{B_2 - B_1}{v_4 - v_1} = C_1, \ \&c.$$

then
$$\begin{aligned} b &= u - A.v + B.vv_1 - C.vv_1v_2 + \&c. \\ b_1 &= A - B(v + v_1) + C(vv_1 + vv_2 + v_1v_2) - \&c. \\ b_2 &= B - C(v + v_1 + v_2) + \&c. \end{aligned}$$

and
$$\begin{aligned} u_x = u &+ A(v_x - v) + B(v_x - v)(v_x - v_1) \\ &+ C(v_x - v)(v_x - v_1)(v_x - v_2) + \&c. \qquad (1) \end{aligned}$$

The value of u_x may likewise be put under the form

$$\begin{aligned} u_x = {} & \frac{(v_x - v_1)(v_x - v_2)\ldots(v_x - v_n)}{(v - v_1)(v - v_2)\ldots(v - v_n)}.u + \frac{(v_x - v)(v_x - v_2)\ldots(v_x - v_n)}{(v_1 - v)(v_1 - v_2)\ldots(v_1 - v_n)}.u_1 \\ & + \frac{(v_x - v)(v_x - v_1)\ldots(v_x - v_n)}{(v_2 - v)(v_2 - v_1)\ldots(v_2 - v_n)}.u_2 + \&c. \\ & + \frac{(v_x - v)(v_x - v_1)\ldots(v_x - v_{n-1})}{(v_n - v)(v_n - v_1)\ldots(v_n - v_{n-1})}u_n. \end{aligned}$$

(7.) Let $v_1 - v = v_2 - v_1 = \&c. = Dv$, then $v_x = v + xDv$,

$$\text{and } A = \frac{D_v u}{Dv}, \qquad A_1 = \frac{D_v u_1}{Dv}, \ \&c.$$

$$B = \frac{1}{1.2} \cdot \frac{D_v^2 u}{Dv^2}, \qquad B_1 = \frac{1}{1.2} \cdot \frac{D_v^2 u_1}{Dv^2}, \ \&c.$$

$$C = \frac{1}{1.2.3} \cdot \frac{D_v^3 u}{Dv^3}, \qquad C_1 = \&c.$$

by substituting these values in (1) we obtain

$$u_x = u + \frac{v_x - v}{1.Dv} \cdot D_v u + \frac{(v_x - v)(v_x - v_1)}{1.2.Dv^2} D_v^2 u + \&c.$$

$$\text{or} \qquad u_x = u + \frac{x}{1} D_v u + \frac{x(x-1)}{1.\ 2} D_v^2 u + \&c.$$

(8.) If $v_x = x$, and $n = 2m$, let the values of u be

$$u_{-m}, \ u_{-m-1}, \ \&c. \ u_{-1}, \ u, \ u_1, \ \&c. \ u_{m-1}, \ u_m,$$

$$\begin{aligned} \text{then } u_x = u &+ \frac{x}{1} \cdot \tfrac{1}{2}(\Delta_x u + \Delta_x u_{-1}) + \frac{x(x^2-1^2)}{1.2.3} \cdot \tfrac{1}{2}(\Delta_x^3 u_{-1} + \Delta_x^3 u_{-2}) \\ &+ \frac{x(x^2-1^2)(x^2-2^2)}{1.2.3.4.5} \cdot \tfrac{1}{2}(\Delta_x^5 u_{-2} + \Delta_x^5 u_{-3}) + \&c. \\ &+ \frac{x^2}{1.2} \cdot \Delta_x^2 u_{-1} + \frac{x^2(x^2-1^2)}{1.2.3.4} \cdot \Delta_x^4 u_{-2} \\ &+ \frac{x^2(x^2-1^2)(x^2-2^2)}{1.2.3.4.5.6} \Delta_x^6 u_{-3} + \&c. \end{aligned}$$

If $n = 2m - 1$, let the values of u be

$$u_{-2m+1}, \ u_{-2m+3}, \ \&c. \ u_{-1}, \ u_1, \ \&c. \ u_{2m-3}, \ u_{2m-1},$$

$$\begin{aligned} \text{then } u_x = \tfrac{1}{2}(u_1 + u_{-1}) &+ \frac{x^2 - 1^2}{2.4} \cdot \tfrac{1}{2}(\Delta_x^2 u_{-1} + \Delta_x^2 u_{-3}) \\ &+ \frac{(x^2-1^2)(x^2-3^2)}{2.4.6.8} \cdot \tfrac{1}{2}(\Delta_x^4 u_{-3} + \Delta_x^4 u_{-5}) + \&c. \\ + \frac{x}{2} \Delta_x u_{-1} + \frac{x(x^2-1^2)}{2.4.6} \Delta_x^3 u_{-3} &+ \frac{x(x^2-1^2)(x^2-3^2)}{2.4.6.8.10} \Delta_x^5 u_{-5} + \&c. \end{aligned}$$

(*Tr. L.* App. 401—7; *L. C. D.* 898—908.)

(9.) If $u_{x,y}$ is a function of two variables, then

$$u_{x,y} = u + \frac{x}{1}\Delta_x u + \frac{x(x-1)}{1\,.\,2}\Delta_x^2 u + \frac{x(x-1)(x-2)}{1\,.\,2\,.\,3}\Delta_x^3 u$$

$$+ \frac{y}{1}\Delta_y u + \frac{x.y}{1.1}\Delta_x\Delta_y u + \frac{x(x-1).y}{1\,.\,2\,.1}\Delta_x^2\Delta_y u$$

$$+ \frac{y(y-1)}{1\,.\,2}\Delta_y^2 u + \frac{x.y(y-1)}{1.1\,.\,2}\Delta_x\Delta_y^2 u$$

$$+ \frac{y(y-1)(y-2)}{1\,.\,2\,.\,3}\Delta_y^3 u + \&c. \text{ [*]}$$

(*L. C. D.* 914, 5.)

APPLICATION OF THE CALCULUS OF DIFFERENCES TO TRIGONOMETRY.

(10.) *Differences of the trigonometrical lines.*

$$D_x \sin x = 2 \sin \tfrac{1}{2} Dx . \cos (x + \tfrac{1}{2} Dx).$$

$$D_x \cos x = -2 \sin \tfrac{1}{2} Dx . \sin (x + \tfrac{1}{2} Dx).$$

$$D_x \tan x = \frac{\sin Dx}{\cos x . \cos (x + Dx)}.$$

$$D_x \cot x = \frac{-\sin Dx}{\sin x . \sin (x + Dx)}.$$

$$D_x (\sin x)^2 = \sin Dx . \sin (2x + Dx).$$

$$D_x (\cos x)^2 = -\sin Dx . \sin (2x + Dx).$$

$$D_x (\tan x)^2 = \frac{\sin Dx . \sin (2x + Dx)}{(\cos x)^2 . \{\cos (x + Dx)\}^2}.$$

$$D_x (\cot x)^2 = -\frac{\sin Dx . \sin (2x + Dx)}{(\sin x)^2 . \{\sin (x + Dx)\}^2}.$$

(*L. C. D.* 892; *Cagn. Trig.* 212—33.)

---

[*] $u_{x,y} = u + \overset{n}{\mathrm{S}}_m \overset{m+1}{\mathrm{S}}_r \, \underline{|x}_{\,m-r+1} . \underline{|y}_{\,r-1} . \Delta_x^{m-r+1} : \Delta_y^{r-1} : u.$

(11.) *The variation of triangles.* Let X, Y, be respectively functions of x, y, any parts of a triangle, and let $X = mY$; then from a given error in one, to determine the corresponding error in the other, we have

$$Dx.d_xX + \frac{(Dx)^2}{1.2} d_x^2X + \frac{(Dx)^3}{1.2.3} d_x^3X + \&c.$$

$$= m\{Dy.d_yY + \frac{(Dy)^2}{1.2}.d_y^2Y + \&c.\ [\beta]$$

All terms except the first may in most cases be neglected, then $\quad Dx.d_xX = mDy.d_yY.$

If great accuracy be requisite, the relation between Dx and Dy may be determined by the solution of the quadratic equation

$$Dx.d_xX + \tfrac{1}{2}(Dx)^2.d_x^2X = m\{Dy.d_yY + \tfrac{1}{2}(Dy)^2.d_y^2Y\}.$$

(*Woodh. Trig.* Ch. xiii.)

(12.) *Corresponding variations of plane triangles.*

[1] Let A, c be invariable; then $DB = -DC$, and

$$Db : \sin DB :: a : \sin(C + DC),$$

$$:: a + Da : \sin C; \qquad d_bB = \frac{\sin C}{a}.$$

$$\tfrac{1}{2}Da : \tan\tfrac{1}{2}DB :: a + \tfrac{1}{2}Da : \tan(C + \tfrac{1}{2}DC),$$

$$\tfrac{1}{2}Da : \sin\tfrac{1}{2}DB :: a : \frac{\sin(C + DC)}{\cos(C + \frac{1}{2}DC)}; \qquad d_aB = \frac{\tan C}{a}.$$

$$Da : Db :: \cos(C + \tfrac{1}{2}DC) : \cos\tfrac{1}{2}DC; \qquad d_ba = \cos C.$$

[2] Let A, a, be invariable; then $DB = -DC$, and

$$\tfrac{1}{2}Dc : \sin\tfrac{1}{2}DC :: c : \frac{\sin C}{\cos(C + \frac{1}{2}DC)}; \qquad d_cC = \frac{\tan C}{c}.$$

$$Db : -Dc :: \cos(B + \tfrac{1}{2}DB) : \cos(C + \tfrac{1}{2}DC); \quad d_cb = -\frac{\cos C}{\cos B}.$$

[β] $\overset{\infty}{S}_n(Dx)^{m-1}.d_x^{m-1}:X = m\{\overset{\infty}{S}_r(Dy)^{r-1}.d_y^{r-1}:Y\}.$

[3] Let b, c, be invariable; then

$\tan\frac{1}{2}DB : \tan\frac{1}{2}DC :: \tan(B+\frac{1}{2}DB) : \tan(C+\frac{1}{2}DC)$;

$$d_B C = \frac{\tan C}{\tan B}.$$

$-\frac{1}{2}Da : \tan\frac{1}{2}DB :: a+\frac{1}{2}Da : \cot(C+\frac{1}{2}DC)$;

$$d_a B = -\frac{\cot C}{a}.$$

$-\sin\frac{1}{2}DA : \sin\frac{1}{2}DB :: a+\frac{1}{2}Da : b.\cos(C+\frac{1}{2}DC)$;

$$d_A B = -\frac{b}{a}\cos C.$$

$\sin\frac{1}{2}DA : \frac{1}{2}Da :: \cos\frac{1}{2}DB : b.\sin(C+\frac{1}{2}DC)$; $d_a A = \frac{1}{b.\sin C}$.

(*Cagn. Trig.* 632—67.)

(13.) *Corresponding variations of spherical triangles.*

[1] Let A, c, be invariable; then

$\sin Db : \sin DB :: \sin(a+Da) : \sin C :: \sin a : \sin(C+DC)$,

$:: \sin a.\sin(a+Da) : \sin c.\sin A$,

$:: \sin c.\sin A : \sin C.\sin(C+DC)$; $d_b B = \frac{\sin C}{\sin a}$.

If A, or $c = 90°$, then

$\sin Db : \sin DB :: \sin b.\cos(b+Db) : \sin B.\cos(B+DB)$;

$$d_b B = \frac{\sin 2B}{\sin 2b}.$$

$\tan\frac{1}{2}Db : -\sin\frac{1}{2}DC :: \tan(a+\frac{1}{2}Da) : \sin(C+\frac{1}{2}DC)$;

$$d_b C = -\frac{\sin C}{\tan a}.$$

$\sin\frac{1}{2}Da : \tan\frac{1}{2}DB :: \sin(a+\frac{1}{2}Da) : \tan(C+\frac{1}{2}DC)$;

$$d_a B = \frac{\tan C}{\sin a}.$$

If $A = 90°$, then

$$\sin \tfrac{1}{2}Db \;:\; -\tfrac{1}{2}\sin DC \;::\; \frac{\sin b}{\cos(b + \frac{1}{2}Db)} \;:\; \sin(C + DC) . \cos C;$$

$$d_b C = -\frac{\sin 2C}{2 \tan b}:$$

$$\tfrac{1}{2}\sin Da \;:\; \sin \tfrac{1}{2}DB \;::\; \sin(a + Da) \cos a \;:\; \frac{\cos B}{\sin(B + \frac{1}{2}DB)};$$

$$d_a B = \frac{2 \cot B}{\sin 2a}.$$

If $c = 90°$, then

$$\sin \tfrac{1}{2}Db \;:\; \tfrac{1}{2}\sin DC \;::\; \frac{\cos b}{\sin(b + \frac{1}{2}Db)} \;:\; \sin(C + DC) \cos C;$$

$$d_b C = \frac{\sin 2C}{2 \cot b}:$$

$$-\tfrac{1}{2}\sin Da \;:\; \sin \tfrac{1}{2}DB \;::\; \sin(a + Da) \cos a \;:\; \frac{\sin B}{\cos(B + \frac{1}{2}DB)};$$

$$d_a B = -\frac{2 \tan B}{\sin 2a}.$$

$$\tan \tfrac{1}{2}Da \;:\; -\tan \tfrac{1}{2}DC \;::\; \tan(a + \tfrac{1}{2}Da) \;:\; \tan(C + \tfrac{1}{2}DC);$$

$$d_a C = \frac{\tan C}{\tan a}.$$

If $A = 90°$, then $d_a C = -\dfrac{\cot B}{\sin a}$; if $c = 90°$, $d_a C = \dfrac{\tan B}{\sin a}$.

$\tan \tfrac{1}{2}Db : \tan \tfrac{1}{2}Da :: \cos \tfrac{1}{2}DC : \cos(C + \tfrac{1}{2}DC)$; $d_b a = \cos C$.

$$\tan \tfrac{1}{2}DB \;:\; -\tan \tfrac{1}{2}DC \;::\; \cos \tfrac{1}{2}DA \;:\; \cos(a + \tfrac{1}{2}Da);$$

$$d_B C = -\cos a.$$

If $A = 90°$, then

$$\sin \tfrac{1}{2}Db \;:\; \sin \tfrac{1}{2}Da \;::\; \frac{\cos b}{\sin(b + \frac{1}{2}Db)} \;:\; \frac{\cos a}{\sin(a + \frac{1}{2}Da)};$$

$$d_b a = \frac{\cot a}{\cot b}.$$

$$\sin\tfrac{1}{2}DB : -\sin\tfrac{1}{2}DC :: \frac{\sin B}{\cos(B+\frac{1}{2}DB)} : \frac{\cos C}{\sin(C+\frac{1}{2}DC)};$$

$$d_B C = -\frac{\cot C}{\tan B}.$$

If $c=90°$, then

$$\sin\tfrac{1}{2}Db : -\sin\tfrac{1}{2}Da :: \frac{\sin b}{\cos(b+\frac{1}{2}Db)} : \frac{\cos a}{\sin(a+\frac{1}{2}Da)};$$

$$d_b a = -\frac{\cot a}{\tan b}.$$

$$\sin\tfrac{1}{2}DB : \sin\tfrac{1}{2}DC : \frac{\cos B}{\sin(B+\frac{1}{2}DB)} : \frac{\cos C}{\sin(C+\frac{1}{2}DC)};$$

$$d_B C = \frac{\cot C}{\tan B}.$$

[2] Let A, a, be invariable; then

$$-\tan\tfrac{1}{2}Db : \tan\tfrac{1}{2}Dc :: \frac{\cos(B+\frac{1}{2}DB)}{\cos\frac{1}{2}DB} : \frac{\cos(C+\frac{1}{2}DC)}{\cos\frac{1}{2}DC};$$

$$d_b c = -\frac{\cos C}{\cos B}.$$

$$-\tan\tfrac{1}{2}DB : \tan\tfrac{1}{2}DC :: \frac{\cos(b+\frac{1}{2}Db)}{\cos\frac{1}{2}Db} : \frac{\cos(c+\frac{1}{2}Dc)}{\cos\frac{1}{2}Dc};$$

$$d_B C = -\frac{\cos c}{\cos b}.$$

$$\sin\tfrac{1}{2}Dc : \sin\tfrac{1}{2}DC :: \frac{\sin c}{\cos(c+\frac{1}{2}Dc)} : \frac{\sin C}{\cos(C+\frac{1}{2}DC)};$$

$$d_c C = \frac{\tan C}{\tan c}.$$

$$d_c B = -\frac{\cos b . \tan C}{\sin c}.$$

If $A=90^\circ$, then

$$-\sin\tfrac{1}{2}Db \;:\; \sin\tfrac{1}{2}Dc \;::\; \frac{\cos b}{\sin(b+\frac{1}{2}Db)} \;:\; \frac{\cos(c+Dc)}{\sin(c+\frac{1}{2}Dc)};$$

$$d_b c = -\frac{\cot c}{\cot b}.$$

$$-\sin DB \;:\; \sin DC \;::\; \cos b \;:\; \cos(c+Dc); \qquad d_B C = -\frac{\cos c}{\cos b}.$$

$$\tfrac{1}{2}\sin Dc \;:\; -\sin\tfrac{1}{2}DB \;::\; \sin c . \cos(c+Dc) \;:\; \frac{\cos B}{\sin(B+\frac{1}{2}DB)};$$

$$d_c B = -\frac{2\cot B}{\sin 2c}.$$

If $a=90^\circ$, then

$$-\sin Db \;:\; \sin Dc \;::\; \cos B \;:\; \cos(C+DC); \qquad d_b c = -\frac{\cos C}{\cos B}.$$

$$-\sin\tfrac{1}{2}DB \;:\; \sin\tfrac{1}{2}DC \;::\; \frac{\cos B}{\sin(B+\frac{1}{2}DB)} \;:\; \frac{\cos(C+DC)}{\sin(C+\frac{1}{2}DC)};$$

$$d_B C = -\frac{\cot C}{\cot B}.$$

$$\sin\tfrac{1}{2}Dc \;:\; -\tfrac{1}{2}\sin DB \;::\; \frac{\cos c}{\sin(c+\frac{1}{2}Dc)} \;:\; \sin B . \cos(B+DB);$$

$$d_c B = -\frac{\sin 2B}{2\cot c}.$$

[3] Let b, c, be invariable; then

$$\tan\tfrac{1}{2}DB \;:\; \tan\tfrac{1}{2}DC \;::\; \tan(B+\tfrac{1}{2}DB) \;:\; \tan(C+\tfrac{1}{2}DC);$$

$$d_B C = \frac{\tan C}{\tan B}.$$

$$-\sin\tfrac{1}{2}Da \;:\; \tan\tfrac{1}{2}DB \;::\; \sin(a+\tfrac{1}{2}Da) \;:\; \cot(C+\tfrac{1}{2}DC);$$

$$d_a B = -\frac{\cot C}{\sin a}.$$

$$\sin\tfrac{1}{2}DA : \sin\tfrac{1}{2}Da :: \sin(a+\tfrac{1}{2}Da) : \sin b . \sin c . \sin(A+\tfrac{1}{2}DA);$$

$$d_a A = \frac{1}{\sin c . \sin B}.$$

$$-\sin\tfrac{1}{2}DA : \tan\tfrac{1}{2}DB :: \frac{\{\sin(a+\frac{1}{2}Da)\}^2}{\sin a.\sin b} : \frac{\sin(A+\frac{1}{2}DA).\sin C}{\sin A.\tan(C+\frac{1}{2}DC)};$$

$$d_B A = \frac{-\sin a}{\sin b.\cos C}.$$

If $c = 90^\circ$, then

$$\tfrac{1}{2}\sin Da : \sin\tfrac{1}{2}DC :: \sin(a+Da).\cos a : \frac{\cos C}{\sin(C+\frac{1}{2}DC)};$$

$$d_a C = \frac{2\cot C}{\sin 2a}:$$

$$\sin\tfrac{1}{2}DA : \sin\tfrac{1}{2}Da :: \frac{\cos a}{\sin(A+\frac{1}{2}DA)} : \frac{\cos a}{\sin(a+\frac{1}{2}Da)};$$

$$d_a A = \frac{\cot A}{\cot a}:$$

$$\sin\tfrac{1}{2}DA : \tfrac{1}{2}\sin DB :: \frac{\sin A}{\cos(A+\frac{1}{2}DA)} : \sin B.\cos(B+DB);$$

$$d_B A = \frac{2\tan A}{\sin 2B}.$$

$$-\sin DA : \sin DC :: \sin(a+Da) : \cos B; \quad d_c A = -\frac{\cos B}{\sin a}.$$

[4] Let B, C, be invariable, then

$$\tan\tfrac{1}{2}Db : \tan\tfrac{1}{2}Dc : \tan(b+\tfrac{1}{2}Db) : \tan(c+\tfrac{1}{2}Dc); \quad d_b c = \frac{\tan c}{\tan b}.$$

$$\sin\tfrac{1}{2}DA : \tan\tfrac{1}{2}Db : \sin(A+\tfrac{1}{2}DA) : \cot(c+\tfrac{1}{2}Dc); \quad d_b A = \frac{\sin A}{\cot c}.$$

$$\sin\tfrac{1}{2}DA : \sin\tfrac{1}{2}Da : \sin(a+\tfrac{1}{2}Da) : \frac{\sin(A+\frac{1}{2}DA)}{\sin B.\sin C};$$

$$d_a A = \sin B.\sin c.$$

$$\sin\tfrac{1}{2}Da : \tan\tfrac{1}{2}Db : \frac{\{\sin(A+\frac{1}{2}DA)\}^2}{\sin A.\sin B} : \frac{\sin(a+\frac{1}{2}Da).\sin c}{\sin a.\tan(c+\frac{1}{2}Dc)};$$

$$d_a b = \frac{\sin B.\cos c}{\sin A}.$$

If $C=90^\circ$, then

$$\tfrac{1}{2}\sin DA \ : \ \sin\tfrac{1}{2}Dc \ :: \ \sin(A+DA)\cos A \ : \ \frac{\cos b}{\sin(b+\frac{1}{2}Db)};$$

$$d_b A=\frac{\sin 2A}{2\cot b}.$$

$$\sin\tfrac{1}{2}DA \ : \ \sin\tfrac{1}{2}Da \ :: \ \frac{\cos A}{\sin(A+\frac{1}{2}DA)} \ : \ \frac{\cos a}{\sin(a+\frac{1}{2}Da)};$$

$$d_a A=\frac{\cot A}{\cot a}.$$

$$\sin\tfrac{1}{2}Da \ : \ \tfrac{1}{2}\sin Db \ :: \ \frac{\sin a}{\cos(a+\frac{1}{2}Da)} \ : \ \sin b\,.\cos(b+Db);$$

$$d_a b=\frac{\sin 2b}{2\tan a}.$$

$$\sin Da \ : \ \sin Dc \ :: \ \sin(a+Da)\,.\cos a \ : \ \sin(c+Dc)\cos c;$$

$$d_a c=\frac{\sin 2c}{\sin 2a}.$$

(*Cagnoli, Trig.* 1213—1420.)

CONSTRUCTION OF LOGARITHMIC AND TRIGONOMETRICAL TABLES.

(14.) *Logarithmic series.*

$$D_x\log_e x=\frac{Dx}{x}-\tfrac{1}{2}\left(\frac{Dx}{x}\right)^2+\tfrac{1}{3}\left(\frac{Dx}{x}\right)^3-\&c. \ ^{\alpha}$$

$$D_x^2\log_e x=-\left\{\left(\frac{Dx}{x}\right)^2-2\left(\frac{Dx}{x}\right)^3+\&c.\right\}. \ ^{\beta}$$

[α] $D_x\log_e x=\overset{\infty}{\mathrm{S}}_m\frac{(-1)^{m-1}}{m}\cdot\left(\frac{Dx}{x}\right)^m.$

[β] $D_x^2\log_e x=2\overset{\infty}{\mathrm{S}}_m(-1)^m\cdot\frac{2^m-1}{m+1}\left(\frac{Dx}{x}\right)^{m+1}.$

$$D_x^3 \log_e x = 2\left(\frac{Dx}{x}\right)^3 - \&c.$$

If Dx is small, these terms will be sufficient; the following series are however still more convergent:

$$D_x \log_e x = 2\left\{\frac{Dx}{2x+Dx} + \frac{1}{3}\left(\frac{Dx}{2x+Dx}\right)^3 + \frac{1}{5}\left(\frac{Dx}{2x+Dx}\right)^5 + \&c.\right\}. \quad ^{\alpha}$$

$$D_x^2 \log_e x = -2\left\{\frac{1}{2}\left(\frac{Dx}{x+Dx}\right)^2 + \frac{1}{4}\left(\frac{Dx}{x+Dx}\right)^4 + \frac{1}{6}\left(\frac{Dx}{x+Dx}\right)^6 + \&c.\right\}. \quad ^{\beta}$$

Two terms of the first series, and the first term of the second, will in most cases be sufficient: if $x = 10000$, and $Dx = 1$, the common logarithms will be exact to fifteen decimal places.

$$D_x \log_e^{-1} x = \log_e^{-1} x \left\{\frac{Dx}{1} + \frac{(Dx)^2}{1.2} + \frac{(Dx)^3}{1.2.3} + \&c.\right\}. \quad ^{\gamma}$$

(*Cagn. Trig.* 372—91; *L. C. D.* 889, 90.)

(15.) The sines of arcs for intervals of a degree having been found by preceding methods, (*Trig.* 42—5.) the sines for minutes may be more easily found by differences: any two differences having been found, the others may be determined by the equation

$$D_x^n \sin x = -\left(2 \sin \tfrac{1}{2} Dx\right)^2 . \left\{D_x^{n-1} \sin x + D_x^{n-2} \sin x\right\}.$$

(16.) The series of differences for any given intervals having been found, the differences for smaller intervals may be found by the following general method: let x be the larger interval, y the smaller, and let $x = py$, then

---

$^{\alpha}$ $D_x \log_e x = 2 \overset{\infty}{\mathrm{S}}_m \frac{1}{2m-1} . \left(\frac{Dx}{2x+Dx}\right)^{2m-1}.$

$^{\beta}$ $D_x^2 \log_e x = -2 \overset{\infty}{\mathrm{S}}_m \frac{1}{2m} \left(\frac{Dx}{x+Dx}\right)^{2m}.$

$^{\gamma}$ $D_x \log_e^{-1} x = \log_e^{-1} x . \overset{\infty}{\mathrm{S}}_m \frac{(Dx)^m}{\underline{|m}}.$

$$D_y = \frac{1}{p} D_x - \frac{p-1}{2p^2} D_x^2 + \frac{(p-1)(2p-1)}{6p^3} D_x^3 - \frac{(p-1)(2p-1)(3p-1)}{24p^4} D_x^4 + \frac{(p-1)(2p-1)(3p-1)(4p-1)}{120p^5} D_x^5 - \&c.$$

$$D_y^2 = \frac{1}{p^2} D_x^2 - \frac{p-1}{p^3} D_x^3 + \frac{(p-1)(11p-7)}{12p^4} D_x^4 - \frac{(p-1)(2p-1)(5p-3)}{12p^5} D_x^5 + \&c.$$

$$D_y^3 = \frac{1}{p^3} D_x^3 - \frac{3(p-1)}{2p^4} D_x^4 + \frac{(p-1)(7p-5)}{4p^5} D_x^5 - \&c.$$

$$D_y^4 = \frac{1}{p^4} D_x^4 - \frac{2(p-1)}{p^5} D_x^5 + \&c.$$

$$D_y^5 = \frac{1}{p^5} D_x^5 - \&c.$$

these differences, which are given as far as the 5th order, are quite sufficient for the calculation of tables.

(17.) $$D_x^n u = \frac{\Delta^n 0^n}{1.2\ldots n} d_x^n u . (Dx)^n + \frac{\Delta^n 0^{n+1}}{1.2\ldots(n+1)} d_x^{n+1} u , (Dx)^{n+1} + \&c.$$ [γ]

In this series, the values of $d_x^n u$, &c. and Dx must be determined in each particular case: for intervals of 1′,

$$Dx = 0{,}000290888208665721596, \&c.$$

The values of $\frac{\Delta^n 0^m}{1.2\ldots m}$ will be the same in every case; these for all values of m and n, from 1 to 12 inclusive, are given in the annexed table:

[γ] $D_x^n u = \overset{\infty}{\mathrm{S}}_m \frac{\Delta^n 0^{n+m-1}}{\underline{|n+m-1}} (Dx)^{n+m-1} . d_x^{n+m-1} u.$

Table of the values of $\frac{\Delta^n 0^m}{1.2\ldots m}$, m and n being $\not> 12$.

	$\frac{0}{1}$	$\frac{0^2}{1.2}$	$\frac{0^3}{1.2.3}$	$\frac{0^4}{1\ldots4}$	$\frac{0^5}{1\ldots5}$	$\frac{0^6}{1\ldots6}$	$\frac{0^7}{1\ldots7}$	$\frac{0^8}{1\ldots8}$	$\frac{0^9}{1\ldots9}$	$\frac{0^{10}}{1\ldots10}$	$\frac{0^{11}}{1\ldots11}$	$\frac{0^{12}}{1\ldots12}$
Δ	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{24}$	$\frac{1}{120}$	$\frac{1}{720}$	$\frac{1}{5040}$	$\frac{1}{40320}$	$\frac{1}{362880}$	$\frac{1}{3628800}$	$\frac{1}{39916800}$	$\frac{1}{479001600}$
Δ^2	0	1	1	$\frac{7}{12}$	$\frac{1}{4}$	$\frac{31}{360}$	$\frac{1}{40}$	$\frac{127}{20160}$	$\frac{17}{12096}$	$\frac{73}{259200}$	$\frac{31}{604800}$	$\frac{2047}{239500800}$
Δ^3	0	0	1	$\frac{3}{2}$	$\frac{5}{4}$	$\frac{3}{4}$	$\frac{43}{120}$	$\frac{23}{160}$	$\frac{605}{12096}$	$\frac{311}{20160}$	$\frac{2591}{604800}$	$\frac{437}{403200}$
Δ^4	0	0	0	1	2	$\frac{13}{6}$	$\frac{5}{3}$	$\frac{81}{80}$	$\frac{37}{72}$	$\frac{6821}{30240}$	$\frac{265}{3024}$	$\frac{55591}{1814400}$
Δ^5	0	0	0	0	1	$\frac{5}{2}$	$\frac{10}{3}$	$\frac{25}{8}$	$\frac{331}{144}$	$\frac{45}{32}$	$\frac{2243}{3024}$	$\frac{1045}{3024}$
Δ^6	0	0	0	0	0	1	3	$\frac{19}{4}$	$\frac{21}{4}$	$\frac{1087}{240}$	$\frac{259}{80}$	$\frac{30083}{15120}$
Δ^7	0	0	0	0	0	0	1	$\frac{7}{2}$	$\frac{77}{12}$	$\frac{49}{6}$	$\frac{1939}{240}$	$\frac{4763}{720}$
Δ^8	0	0	0	0	0	0	0	1	4	$\frac{25}{3}$	12	$\frac{4819}{360}$
Δ^9	0	0	0	0	0	0	0	0	1	$\frac{9}{2}$	$\frac{21}{2}$	$\frac{135}{8}$
Δ^{10}	0	0	0	0	0	0	0	0	0	1	5	$\frac{155}{12}$
Δ^{11}	0	0	0	0	0	0	0	0	0	0	1	$\frac{11}{2}$
Δ^{12}	0	0	0	0	0	0	0	0	0	0	0	1

(18.) If the log. sines be calculated for large intervals, as for every 10°, the differences for every degree may be thus found:

$$D_x \log_\epsilon \sin x = 2\left\{\frac{\sin(x+Dx)-\sin x}{\sin(x+Dx)+\sin x}\right.$$

$$\left. + \tfrac{1}{3}\left(\frac{\sin(x+Dx)-\sin x}{\sin(x+Dx)+\sin x}\right)^3 + \&c.\right\},$$

$$= 2\left\{\frac{D_x \sin x}{2\sin x + D_x \sin x} + \tfrac{1}{3}\left(\frac{D_x \sin x}{2\sin x + D_x \sin x}\right)^3 + \&c.\right\}$$

$$D_x^2 \log_\epsilon \sin x = -2\left\{\frac{(\sin Dx)^2}{(\cos Dx)^2 + \cos(2x+Dx)}\right.$$

$$\left. + \tfrac{1}{3}\left(\frac{(\sin Dx)^2}{(\cos Dx)^2 + \cos(2x+Dx)}\right)^3 + \&c.\right\};$$

this series converges very rapidly.

(*Enc. Met. Trig.* 194—209; *L. C. D.* 893—6.)

INVERSE METHOD OF DIFFERENCES.

(19.) *Integration of algebraical functions.*

$\Sigma_x \Delta_x u_x = u_x + const.$

$\Sigma_x a . u_x = a\Sigma_x u_x$; if a is either independent of x, or such a function of x, that $a_x = a_{x+1}$.

$\Sigma_x\{{}^1u_x + {}^2u_x + \&c. + {}^nu_x\} = \Sigma_x {}^1u_x + \Sigma_x {}^2u_x + \&c. + \Sigma_x {}^nu_x.$

$\Sigma_x(u_x . \Delta_x v_x) = u_x . v_x - \Sigma_x v_{x+1} . \Delta_x u_x.$

Let $u_x = a + bx$, then

$$\Sigma_x(u_x . u_{x+1} \ldots u_{x+n-1}) = \frac{u_{x-1} . u_x . u_{x+1} \ldots u_{x+n-1}}{(n+1)b};$$

$$\Sigma_x \frac{1}{u_x . u_{x+1} \ldots u_{x+n-1}} = \frac{-1}{(n-1)b . u_x . u_{x+1} \ldots u_{x+n-2}}:$$

The fraction $\dfrac{A_n x^n + A_{n-1} x^{n-1} + \&c. + A_1 x + A}{u_x . u_{x+1} \ldots \ldots u_{x+n+1}}$, in which the numerator is at least two dimensions lower than the denominator, may be decomposed into integrable fractions by assuming

$$A_n x^n + A_{n-1} x^{n-1} + \text{\&c.} + A_1 x + A$$
$$= B + B_1 . u_x + B_2 . u_x . u_{x+1} + \text{\&c.} + B_n . u_x . u_{x+1} \ldots u_{x+n-1},$$

developing the latter quantity in powers of x, and equating coefficients.

(20.) $\Sigma_x 1 = x + \mathit{const.}$

$$\Sigma_x x = \tfrac{1}{2} x(x-1) + \mathit{const.}$$
$$\Sigma_x x^2 = \tfrac{1}{2} . \tfrac{1}{3} . x(x-1)(2x-1) + \mathit{const.}$$
$$\Sigma_x x^3 = \tfrac{1}{4} x^2 (x-1)^2 + \mathit{const.}$$
$$\text{\&c.} = \quad \text{\&c.}$$

$$\Sigma_x x^m = \frac{x^{m+1}}{m+1} - \tfrac{1}{2} x^m + \frac{1}{2.3} . \frac{m}{2} x^{m-1}$$
$$- \frac{1}{2.3.5} . \frac{m(m-1)(m-2)}{2 \ . \ 3 \ . \ 4} x^{m-3} + \frac{1}{2.3.7} . \frac{m(m-1)\ldots(m-4)}{2 \ . \ 3 \ \ldots \ 6} x^{m-5}$$
$$- \frac{1}{2.3.5} . \frac{m(m-1)\ldots(m-6)}{2 \ . \ 3 \ \ldots \ 8} x^{m-7} + \frac{5}{2.3.11} . \frac{m(m-1)\ldots(m-8)}{2 \ . \ 3 \ \ldots \ 10} x^{m-9}$$
$$- \frac{691}{2.3.5.7.13} . \frac{m \ldots (m-10)}{2 \ldots \ 12} x^{m-11} + \frac{7}{2.3} . \frac{m \ldots (m-12)}{2 \ldots \ 14} x^{m-13}$$
$$- \frac{3617}{2.3.5.17} . \frac{m(m-1) \ldots (m-14)}{2 \ . \ 3 \ \ldots \ 16} x^{m-15} + \text{\&c.} + \mathit{const.}^{a}$$

(*Tr. L.* App. 368—72; *L. C. D.* 943—54; *L. D. C.* 516—23.)

(21.) The numerical coefficients of x^{m-1}, x^{m-3}, &c. are the *numbers of Bernoulli*; let them be represented by $\mathcal{C}_2$, $\mathcal{C}_4$, &c. respectively, then

$$0 = \tfrac{1}{3} - \tfrac{1}{2} + \mathcal{C}_2,$$
$$0 = \tfrac{1}{5} - \tfrac{1}{2} + \tfrac{4}{2} \mathcal{C}_2 + \mathcal{C}_4,$$

---

[a] $\Sigma_x x^m = \dfrac{x^{m+1}}{m+1} - \tfrac{1}{2} x^m + \overset{\infty}{\mathrm{S}}_r \dfrac{\mathcal{C}_{2r}}{\underline{|2r}} . \underline{|m}_{\,2r-1} . x^{m-2r+1}.$

$$0=\frac{1}{7}-\frac{1}{2}+\frac{6}{2}\mathcal{C}_2+\frac{6.5.4}{2.3.4}\mathcal{C}_4+\mathcal{C}_6,$$

&c. &c.

$$0=\frac{1}{m+1}-\frac{1}{2}+\frac{m}{2}\mathcal{C}_2+\frac{m(m-1)(m-2)}{2\ .\ 3\ .\ 4}\mathcal{C}_4$$
$$+\frac{m(m-1)\dots(m-4)}{2\ .\ 3\ \dots\ 6}\mathcal{C}_6+\frac{m(m-1)\dots(m-6)}{2\ .\ 3\ \dots\ 8}\mathcal{C}_8+\&c.$$ [α]

(*Moivre, Miscell. Anal. Supp.* p. 6.)

$\mathcal{C}_{2m}$ is the coefficient of $\frac{t^{2m}}{1.2.3\dots 2m}$ in the expansion of $\frac{t}{\epsilon^t-1}$, or

$$\mathcal{C}_{2m}=(-1)^{m-1}\frac{2m}{2^{2m}(2^{2m}-1)}\{1^{2m-1}-(2^{2m-1}-\frac{2m}{1}1^{2m-1})$$
$$+(3^{2m-1}-\&c.)-\&c.\}.$$ [β]

$$=d_t^{2m}\frac{1}{\epsilon^t-1}.$$ (*Tr. L.* App. 408.)

(22.) *Integration of exponential functions.*

$$\Sigma_x a^x=\frac{a^x}{a-1}+const.$$

$$\Sigma_x\cos x\theta=\frac{\sin(x+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}+const.$$

$$\Sigma_x\sin x\theta=-\frac{\cos(x+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}+const.$$

$$\Sigma_x\cos(a+bx)\theta=\frac{\sin(a+\overline{x+\frac{1}{2}}.b)\theta}{2\sin\frac{1}{2}b\theta}+const.$$

$$\Sigma_x\sin(a+bx)\theta=-\frac{\cos(a+\overline{x+\frac{1}{2}}.b)\theta}{2\sin\frac{1}{2}b\theta}+const.$$

[α] $0=\frac{1}{m+1}-\frac{1}{2}+\overset{\infty}{\mathrm{S}}_r\,\underline{|m}_{\,2r-1}\cdot\frac{\mathcal{C}_{2r}}{\underline{|2r}}.$

[β] $\underline{|2m}.\mathcal{C}_{2m}=\varpi^{2m}a^{-1},\ \{a_{r-1}=\frac{1}{\underline{|r}}\}.$

$$\Sigma_x(a^x.u_x)=\frac{a^x.u_x}{a-1}-\frac{a^{x+1}.\Delta_x u_x}{(a-1)^2}+\frac{a^{x+2}.\Delta_x^2 u_x}{(a-1)^3}-\&c.+const.$$ [α]

Let $b.a^x+c=u_x$, then

$$\Sigma_x a^x.u_x.u_{x+1}\ldots u_{x+n-1}$$

$$=\frac{a}{b(a^{n+1}-1)}.u_{x-1}.u_x.u_{x+1}\ldots u_{x+n-1}+const.;$$

$$\Sigma_x\frac{a^x}{u_x.u_{x+1}\ldots u_{x+n-1}}=const.-\frac{1}{b(a^{n-1}-1)}.\frac{1}{u_x.u_{x-1}\ldots u_{x+n-2}}.$$

(*Tr. L.* App. 373—5; *L. C. D.* 955—60.)

(23.) $\Sigma_x(u_x.v_x)=u_x.\Sigma_x v_x-\Sigma_x(\Delta_x u_x.\Sigma_x v_{x+1}),$

$$=u_x.\Sigma_x v_x-\Delta_x u_x.\Sigma_x^2 v_{x+1}+\Delta_x^2 u_x.\Sigma_x^3 v_{x+2}-\&c.$$

$$+(-1)^{n-1}\{\Delta_x^{n-1}u_x.\Sigma_x^n v_{x+n-1}-\Sigma_x(\Delta_x^n u_x.\Sigma_x^n v_{x+n})\};$$ [β]

$$=u_x\Sigma_x v_x-\Delta_x u_x(\Sigma_x^2 v_x+\Sigma_x v_x)$$

$$+\Delta_x^2 u_x(\Sigma_x^3 v_x+2\Sigma_x^2 v_x+\Sigma_x v_x)-\&c.$$

$$\Sigma_x^2(u_x.v_x)=u_x.\Sigma_x^2 v_x-2\Delta_x u_x.\Sigma_x^3 v_{x+1}+3\Delta_x^2 u_x.\Sigma_x^4 v_{x+2}-\&c.$$

$$+(-1)^{n-1}n\Delta_x^{n-1}u_x.\Sigma_x^{n+1}v_{x+n-1}$$

$$-(-1)^{n-1}\{\Sigma_x^2(\Delta_x^n u_x.\Sigma_x^n v_{x+n})+n\Sigma_x(\Delta_x^n u_x.\Sigma_x^{n+1}v_{x+n})\}.$$

&c. = &c.

$$\Sigma_x^r(u_x.v_x)=u_x.\Sigma_x^r v_x-\frac{r}{1}\Delta_x u_x.\Sigma_x^{r+1}v_{x+1}+\frac{r(r+1)}{1\;.\;2}\Delta_x^2 u_x.\Sigma_x^{r+2}v_{x+2}$$

$$-\&c.+(-1)^{n-1}.\frac{n(n+1)\ldots(n+r-2)}{1\;.\;2\;\ldots\;(r-1)}\Delta_x^{n-1}u_x\Sigma_x^{r+n-1}v_{x+n-1}$$

[α] $\Sigma_x(a^x.u_x)=\overset{\infty}{\text{S}}_m(-1)^{m-1}\frac{a^{x+m-1}}{(a-1)^m}\Delta_x^{m-1}u_x+const.$

[β] $\Sigma_x(u_x.v_x)=\overset{n}{\text{S}}_m(-1)^{m-1}.\Delta_x^{m-1}u_x.\Sigma_x^m v_{x+m-1}$

$+(-1)^n.\Sigma_x(\Delta_x^n u_x.\Sigma_x^n v_{x+n}).$

$$+(-1)^n\{\Sigma_x^r(\Delta_x^n u_x\,.\,\Sigma_x^n v_{x+n})+\frac{n}{1}\,.\,\Sigma_x^{r-1}(\Delta_x^n u_x.\Sigma_x^{n+1}v_{x+n})$$

$$+\,\&\text{c.}+\frac{n(n+1)\ldots(n+r-2)}{1\;.\;2\;\ldots\;(r-1)}\Sigma_x(\Delta_x^n u_x.\Sigma_x^{n+r-1}v_{x+n})\}.\quad{}^{\alpha}$$

(*Tr. L.* App. 376, 7; *L. C. D.* 959—62.)

$$(24.)\quad \Sigma_x u_x=\int_x u_x-\tfrac{1}{2}u_x+\frac{\mathfrak{C}_2}{2}d_x u_x+\frac{\mathfrak{C}_4}{4}.\frac{d_x^3 u_x}{1.2.3}+\&\text{c.}$$

$$=(\epsilon^{d_x}-1)^{-1}u_x.\quad{}^{\beta}$$

$$\Sigma_x^n u_x=(\epsilon^{d_x}-1)^{-n}u_x.\quad{}^{\gamma}$$

$$\Sigma_x(b^x.u_x)=b^x(b.\epsilon^{d_x}-1)^{-1}u_x.\quad{}^{\delta}$$

$$\Sigma_x^n(b^x.u_x)=b^x(b.\epsilon^{d_x}-1)^{-n}.u_x.$$

(*Tr. L.* App. 378; *L. C. D.* 963—80.)

$$^{\alpha}\quad \Sigma_x^r(u_x.v_x)=\overset{n}{\mathrm{S}}_m(-1)^{m-1}.\frac{|\underline{m+r-2}_{\,r-1}}{|\underline{r-1}}\Delta_x^{m-1}u_x.\Sigma_x^{r+m-1}v_{x+m-1}$$

$$+(-1)^n\overset{r}{\mathrm{S}}_s\frac{|\underline{n+s-2}_{\,s-1}}{|\underline{s-1}}\Sigma_x^{r-s+1}(\Delta_x^n u_x.\Sigma_x^{n+s-1}v_{x+n}).$$

$$^{\beta}\quad \Sigma_x u_x=\int_x u_x-\tfrac{1}{2}u_x+\overset{\infty}{\mathrm{S}}_m\frac{\mathfrak{C}_{2m}}{2m}.d_x^{2m-1}:u_x.$$

$$=\overset{\infty}{\mathrm{S}}_m\varpi^{m-1}a^{-1}.d_x^{m-2}u_x,\ \left\{a_{r-1}=\frac{1}{|\underline{r}}\right\}.$$

$$^{\gamma}\quad \Sigma_x^n u_x=\overset{\infty}{\mathrm{S}}_m\varpi^{m-1}a^{-n}.d_x^{m-(n+1)}u_x;\ \left\{a_{r-1}=\frac{1}{|\underline{r}}\right\}.$$

$$^{\delta}\quad \Sigma_x(b^x.u_x)=b^x.\overset{\infty}{\mathrm{S}}_m\varpi^{m-1}a^{-1}.b^{m-2}d_x^{m-2}u_x;\ \left\{a_{r-1}=\frac{1}{|\underline{r}}\right\}.$$

EQUATIONS OF DIFFERENCES.

(25.) The complete integral of an equation of differences of the n^{th} order

$$0 = \phi(x, u_x, u_{x+1}, \text{\&c. } u_{x+n})$$

must contain n independent arbitrary constants.

(26.) The general equation of the first order and degree is

$$u_{x+1} = a_x u_x + b_x.$$

Let $u_x = v_x . a_1 . a_2 \ldots a_{x-1} = v_x . \overset{x-1}{\mathrm{P}}_m (a_m)$, *

then $(v_{x+1} - v_x)\overset{x}{\mathrm{P}}_m (a_m) = b_x$, or $\Delta_x v_x . \overset{x}{\mathrm{P}}_m (a_m) = b_x$;

$$\therefore \; u_x = \overset{x-1}{\mathrm{P}}_m (a_m) . \left\{ \Sigma_x \frac{b_x}{\overset{x}{\mathrm{P}}_m (a_m)} + const. \right\}.$$

(*Tr. L.* App. 379—81; *L. C. D.* 1038.)

(27.) The general equation of the n^{th} order and first degree is

$$u_{x+n} + {}^1a_x . u_{x+n-1} + {}^2a_x . u_{x+n-2} + \text{\&c.} + {}^na_x . u_x = b_x; \qquad (1)$$

the integration of which is reducible to that of the equation

$$u_{x+n} + {}^1a_x . u_{x+n-1} + {}^2a_x . u_{x+n-2} + \text{\&c.} + {}^na_x . u_x = 0: \qquad (2)$$

which latter equation is always integrable if $n-1$ particular integrals can be obtained; and if m particular integrals can be obtained, it may be reduced to an equation of the $(n-m)^{th}$ order. The general method may, to avoid complexity, be illustrated by its application to the equation

$$u_{x+3} + {}^1a_x . u_{x+2} + {}^2a_x . u_{x+1} + {}^3a_x . u_x = b_x.$$

Let this equation be obtained by elimination from the equations

$$\begin{aligned} u_{x+1} + {}^1v_x . u_x &= {}^1u_x, \\ {}^1u_{x+1} + {}^2v_x . {}^1u_x &= {}^2u_x, \qquad (3) \\ {}^2u_{x+1} + {}^3v_x . {}^2u_x &= b_x, \end{aligned}$$

---

* See Appendix.

then
$$\begin{aligned} {}^1a_x &= {}^1v_{x+2} + {}^2v_{x+1} + {}^3v_x, \\ {}^2a_x &= {}^1v_{x+1}.{}^2v_{x+1} + {}^1v_{x+1}.{}^3v_x + {}^2v_x.{}^3v_x, \qquad (4) \\ {}^3a_x &= {}^1v_x.{}^2v_x.{}^3v_x. \end{aligned}$$

From (3),
$$\begin{aligned} u_x &= \overset{x-1}{\mathrm{P}}_m(-{}^1v_m)\left\{{}^1C + \Sigma_x \frac{{}^1u_x}{\overset{x}{\mathrm{P}}_m(-{}^1v_m)}\right\}, \\ {}^1u_x &= \overset{x-1}{\mathrm{P}}_m(-{}^2v_m)\left\{{}^2C + \Sigma_x \frac{{}^2u_x}{\overset{x}{\mathrm{P}}_m(-{}^2v_m)}\right\}, \qquad (5) \\ {}^2u_x &= \overset{x-1}{\mathrm{P}}_m(-{}^3v_m)\left\{{}^3C + \Sigma_x \frac{b_x}{\overset{x}{\mathrm{P}}_m(-{}^3v_m)}\right\}; \end{aligned}$$

in which ${}^1C$, ${}^2C$, and ${}^3C$ are arbitrary constants.

From these equations we may obtain by elimination

$$u_x = {}^1C.{}^1U_x + {}^2C.{}^2U_x + {}^3C.{}^3U_x + V_x,$$

in which
$$\begin{aligned} {}^1U_x &= \overset{x-1}{\mathrm{P}}_m(-{}^1v_m), \\ {}^2U_x &= \overset{x-1}{\mathrm{P}}_m(-{}^1v_m).\Sigma_x \frac{\overset{x-1}{\mathrm{P}}_m(-{}^2v_m)}{\overset{x}{\mathrm{P}}_m(-{}^1v_m)}, \qquad (6) \\ {}^3U_x &= \overset{x-1}{\mathrm{P}}_m(-{}^1v_m).\Sigma_x \left\{\frac{\overset{x-1}{\mathrm{P}}_m(-{}^2v_m)}{\overset{x}{\mathrm{P}}_m(-{}^1v_m)}.\Sigma_x \frac{\overset{x-1}{\mathrm{P}}_m(-{}^3v_m)}{\overset{x}{\mathrm{P}}_m(-{}^2v_m)}\right\}, \end{aligned}$$

$$V_x = \overset{x-1}{\mathrm{P}}_m(-{}^1v_m).\Sigma_x {}_1\left\{\frac{\overset{x-1}{\mathrm{P}}_m(-{}^2v_m)}{\overset{x}{\mathrm{P}}_m(-{}^1v_m)}.\Sigma_x {}_2\left\{\frac{\overset{x-1}{\mathrm{P}}_m(-{}^3v_m)}{\overset{x}{\mathrm{P}}_m(-{}^2v_m)}.\Sigma_x \frac{b_x}{\overset{x}{\mathrm{P}}_m(-{}^3v_m)}\right\}_2\right\}_1.$$

If $b_x = 0$, then $V_x = 0$, and the equation

$$u_x = {}^1C.{}^1U_x + {}^2C.{}^2U_x + {}^3C.{}^3U_x$$

is the complete integral of

$$u_{x+3} + {}^1a_x.u_{x+2} + {}^2a_x.u_{x+1} + {}^3a_x.u_x = 0.$$

If this equation is known, then ${}^1v_x$, ${}^2v_x$, &c. may be determined, for from (6)

$${}^1v_x = \frac{{}^1U_{x+1}}{{}^1U_x},$$

$${}^2v_x = \frac{{}^1U_{x+2}.\Delta_x\left\{\dfrac{{}^2U_{x+1}}{{}^1U_{x+1}}\right\}}{{}^1U_{x+1}.\Delta_x\left\{\dfrac{{}^2U_x}{{}^1U_x}\right\}},$$

$$\&c. = \quad \&c.$$

these quantities being substituted in V_x, and the result added to the complete integral of (2), the complete integral of (1) will be obtained.

If $n-1$ of the quantities U_x are known, the n^{th} may be obtained from the equations (3).

(28.) If the coefficients of (2) are constant, that equation will be satisfied by assuming $u_x = e^x$; then, dividing by e^x, we have

$$e^n + {}^1a.e^{n-1} + {}^2a.e^{n-2} + \&c. + {}^{n-1}a.e + {}^na = 0.$$

Let e_1, e_2, &c. e_n, be the roots of this equation, then

$$u_x = {}^1C.e_1^x + {}^2C.e_2^x + \&c. + {}^nC.e_n^x$$

is the complete integral required.

If any of the quantities e_1, e_2, &c. are equal, or impossible, the changes which take place in the value of u_x are analogous to those in (*Int. Calc.* 43.)

(29.) If we wish to extend the integral of (2) to (1), then

$$u_x = {}^1C.e_1^x + \&c. + {}^nC.e_n^x + \frac{b}{1 + {}^1a + {}^2a + \&c. + {}^na}.$$

(*Tr. L.* App. 382—5; *L. C. D.* 1036—52.)

(30.) An equation of the second order and first degree

$$u_{x+2} + {}^1a_x.u_{x+1} + {}^2a_x.u_x = 0, \qquad (1)$$

may be thus solved: assume

$$u_x = v_x.\overset{x-2}{\mathrm{P}}_m(-{}^1a_m), \qquad \text{and } c_{x+2} = \frac{-{}^2a_x}{{}^1a_x.{}^1a_{x-1}},$$

then $$v_{x+2} - v_{x+1} - c_{x+2}.v_x = 0, \qquad (2)$$

and $$\frac{v_x}{v_{x-1}} = 1 + \frac{c_x}{\frac{v_{x-1}}{v_{x-2}}} = 1 + \frac{c_x}{1+}\,\frac{c_{x-1}}{\frac{v_{x-2}}{v_{x-3}}} = \&c.$$

$$= 1 + \frac{c_x}{1+}\,\frac{c_{x-1}}{1+\&c.\ldots}\,\frac{c_{i+1}}{1+c_i},$$

$\frac{v_{i-1}}{v_{i-2}}$ being constant, and assumed equal to unity. Let this continued fraction be represented by $F(c_x)$, then will $\overset{x}{\mathrm{P}}_m\{F(c_m)\}$ be a particular integral of v_x in (2), and

$$v_x = \overset{x-1}{\mathrm{P}}_m\{F(c_{m+1})\}.\left\{{}^1C + {}^2C.\Sigma_x \overset{x}{\mathrm{P}}_m\left(\frac{-c_{m+1}}{c_{m+1}+F(c_m)}\right)\right\}.$$

${}^1C$ and ${}^2C$ being arbitrary constants: this value of v_x multiplied by $\overset{x-2}{\mathrm{P}}_m(-{}^1a_m)$ will give the required value of u_x. From this the value of u_x in the equation

$$u_{x+2} + {}^1a_x.u_{x+1} + {}^2a_x.u_x = b_x$$

may be obtained: the result is

$$u_x = \overset{x-2}{\mathrm{P}}_m\{-{}^1a_m.F(c_{m+2})\} \times$$

$$\Sigma_x \underset{1}{\Bigg\{} \overset{x}{\mathrm{P}}_m\left(\frac{-c_{m+1}}{c_{m+1}+F(c_m)}\right) \Sigma_x \underset{2}{\Bigg\{} b_x.\overset{x}{\mathrm{P}}_m\left(\frac{c_{m+1}+F(c_m)}{{}^1a_m.c_{m+2}.F(c_m)}\right)\underset{2}{\Bigg\}}\underset{1}{\Bigg\}},$$

in which a constant must be added after each integration.

(31.) The equation of the second degree,

$$u_{x+1}.u_x - au_{x+1} + bu_x + c = 0, \qquad (1)$$

in which the coefficients are constant, may be reduced to

$$v_{x+2} + (a+b)v_{x+1} + (ab+c)v_x = 0, \qquad (2)$$

by assuming $u_x = \frac{v_{x+1}}{v_x} + a$; let the complete integral of (2) be

$$v_x = c_1.e_1^x + c_2.e_2^x,$$

and let $\frac{c_2}{c_1} = k$, then the integral of (1) is

$$u_x = \frac{e_1^{x+1} + k.e_2^{x+1}}{e_1^x + k.e_2^x} + a.$$

If $b = a$, and $c = 1$, then

$$u_x = \tan\{(x + \textit{const.}).\tan^{-1}(-a)^{-1}\}.$$

The equation $u_{x+1}.u_x + a_x.u_{x+1} + b_x.u_x + c_x = 0$, may be reduced to

$$v_{x+2} + (b_x - a_{x+1})v_{x+1} + (c_x - a_x.b_x)v_x = 0, \qquad (1)$$

by assuming $u_x = \frac{v_{x+1}}{v_x} - a_x$: let ${}^1v_x$, ${}^2v_x$, be the two particular integrals of (1), then

$$u_x = \frac{{}^1v_{x+1} + k.{}^2v_{x+1}}{{}^1v_x + k.{}^2v_x} - a_x.$$

(32.) The equation of the third degree

$$u_{x+2}.u_{x+1}.u_x - a(u_{x+2} + u_{x+1} + u_x) = 0,$$

may, by assuming $u_x = \tan v_x.\sqrt{a}$, be reduced to

$$v_{x+2} + v_{x+1} + v_x = 0,$$

the complete integral of which is

$$v_x = c_1.\cos\tfrac{2}{3}\pi.x + c_2.\sin\tfrac{2}{3}\pi.x,$$

then $$u_x = \sqrt{a}.\tan(c_1.\cos\tfrac{2}{3}\pi.x + c_2.\sin\tfrac{2}{3}\pi.x).$$

(*Tr. L.* App. 386; *Mem. Analyt. Soc.* 1813. pp. 84—95.)

EQUATIONS OF MIXED DIFFERENCES.

(33.) Equations of mixed differences, in which u_x, u_{x+1}, &c. and their differential coefficients do not exceed the first degree, and in which the coefficients are constant, may be rendered integrable by assuming $u_x = v_x + k$: the equation

$$u_x + au_{x+1} + bd_xu_x + cd_xu_{x+1} + f = 0,$$

may by assuming $u_x = v_x - \frac{f}{1+a}$ be reduced to

$$v_x + av_{x+1} + bd_xv_x + cd_xv_{x+1},$$

which is satisfied by putting $v_x = e^{kx}$, and e_1, e_2, &c. being the roots of the equation

$$1 + ae^k + bk + cke^k = 0,$$

$$u_x = {}^1c.e_1^x + {}^2c.e_2^x + \&c. - \frac{f}{1+a}.$$

The equation

$$u_x - (1 + a.\epsilon^x)d_xu_x + b(1 + a.\epsilon^{x+1})u_{x+1} + c,$$

may be thus reduced to an integrable form: let $u_x = z_x.v_x$, in which z_x is determined by the differential equation

$$z_x - (1 + a\epsilon^x)d_xz_x = 1;$$

from which we obtain $z_x = \frac{1}{1 + a\epsilon^x}$, then the proposed equation becomes

$$v_x + bv_{x+1} - d_xv_x + c = 0.$$

(34.) The equation

$$d_xu_{x+1} + a_x.d_xu_x + b_x.u_{x+1} + c_x.u_x = e_x$$

may, by assuming $u_{x+1} + a_x.u_x = v_x$, be reduced to

$$d_xv_x + b_x.v_x = e_x + (d_xa_x + a_x.b_x - c_x)u_x,$$

which becomes

$$d_xv_x + b_x.v_x = e_x, \text{ if } d_xa_x + a_x.b_x - c_x = 0:$$

if this latter condition is not satisfied, assume

$$d_xa_x + a_x.b_x - c_x = g_x,$$

$$\text{then } u_x = \frac{d_xv_x + b_x.v_x - e_x}{g_x},$$

$$\text{and } u_{x+1} = \frac{d_xv_{x+1} + b_{x+1}.v_{x+1} - e_{x+1}}{g_{x+1}};$$

$$\text{and if } \frac{a_x.g_{x+1}}{g_x} = {}^1a_x, \qquad \frac{a_x.b_x.g_{x+1}}{g_x} - g_{x+1} = {}^1c_x,$$

$$e_{x+1} + \frac{a_x.e_x.g_{x+1}}{g_x} = {}^1e_x,$$

the equation $u_{x+1} + a_x . u_x = v_x$ becomes

$$d_x v_{x+1} + {}^1a_x . d_x v_x + b_{x+1} . v_{x+1} + {}^1c_x . v_x = {}^1e_x,$$

which is of the same form as the original equation, and with which the same transformation may be repeated, until we arrive at an equation in which

$$d_x a_x + a_x . b_x - c_x = 0,$$

when this condition is satisfied, a value of v_x may be determined from the equation $d_x v_x + b_x . v_x = e_x$,

and from that, by successive substitutions, the value of u_x.

If we begin by assuming

$$d_x u_x + b_{x-1} . u_x = v_x,$$

the above transformations may be effected in an inverse order: the steps of the operation are analogous to the preceding.

The values of x obtained by the first method are

$$u_x = A_x + B_x . c + C_x . p_x, \qquad \text{if } g_x = 0,$$

$$u_x = {}^1A_x + {}^1B_x . c + {}^1C_x . p_x + {}^1E_x . d_x p_x, \qquad \text{if } {}^1g_x = 0;$$

&c. &c.

and those obtained by the second are

$$u_x = A_x + B_x . c + C_x \int_x Q_x . p_x, \qquad \text{if } g_x = 0,$$

$$u_x = {}^1A_x + {}^1B_x . c + {}^1C_x \int_x {}^1Q_x . p_x + {}^1e_x \int_x {}^1R_x . p_x, \qquad \text{if } {}^1g_x = 0;$$

&c. &c.

in which p is any periodic function.

(*Tr. L.* 387, 8; *L. C. D.* 1256—63.)

SUMMATION OF SERIES.

(35.) Let $u_1 + u_2 + \&c. + u_x$ be represented by $\overset{x}{\mathrm{S}}_m u_m$, then

$$\overset{x}{\mathrm{S}}_m u_m = \Sigma_x u_{x+1} + const. = \Sigma_x u_{x+1} - \Sigma_{x=0} u_{x+1}.$$

(36.) $\overset{x}{\mathrm{S}}_n (-1)^{n-1}.n^m$

$$=(-1)^x\left\{\tfrac{1}{2}x^m+\frac{(2^2-1)\,\mathfrak{C}_2}{1.2}m x^{m-1}+\frac{(2^4-1)\,\mathfrak{C}_4}{1.2.3.4}m(m-1)(m-2)x^{m-3}\right.$$

$$\left.+\frac{(2^6-1)\,\mathfrak{C}_6}{1.......6}m(m-1)...(m-4)\,x^{m-5}+\&\text{c.}\right\}+\textit{const.}\qquad[\alpha]$$

$$\overset{x}{\mathrm{S}}_m\frac{1}{am+b}=\frac{1}{a}\log_\epsilon b+\frac{1}{2(ax+b)}-\frac{\mathfrak{C}_2.a}{2\,(ax+b)^2}$$

$$+\frac{\mathfrak{C}_4.a^3}{4(ax+b)^4}-\&\text{c.}+\textit{const.}\qquad[\beta]$$

$$\overset{x}{\mathrm{S}}_r\frac{1}{r^m}=-\frac{1}{(m-1)x^{m-1}}+\frac{1}{2x^m}-\frac{m\mathfrak{C}_2}{2x^{m+1}}$$

$$+\frac{m(m+1)(m+2)\mathfrak{C}_4}{2.3.4x^{m+3}}-\&\text{c.}+\textit{const.}\qquad[\gamma]$$

$$\overset{x}{\mathrm{S}}_m\log_\epsilon m=\tfrac{1}{2}\log_\epsilon 2\pi+(x+\tfrac{1}{2})\log_\epsilon x-x+\frac{\mathfrak{C}_2}{1.2x}-\frac{\mathfrak{C}_4}{3.4x^3}+\&\text{c.}\qquad[\delta]$$

(*L. C. D.* 990—1008.)

[α] $$\overset{x}{\mathrm{S}}_n(-1)^{n-1}.n^m=(-1)^x.\left\{\tfrac{1}{2}x^m+\overset{\infty}{\mathrm{S}}_r\frac{2^{2r}-1}{\lfloor\underline{2r}}\mathfrak{C}_{2r}.\lfloor\underline{m}_{\,2r-1}.x^{m-2r+1}\right\}$$

$$+\textit{const.}$$

[β] $$\overset{x}{\mathrm{S}}_m\frac{1}{am+b}=\frac{1}{a}\log_\epsilon b+\frac{1}{2(ax+b)}+\overset{\infty}{\mathrm{S}}_m(-1)^m\frac{\mathfrak{C}_{2m}.a^{2m-1}}{2m(ax+b)^{2m}}$$

$$+\textit{const.}$$

[γ] $$\overset{x}{\mathrm{S}}_r\frac{1}{r^m}=-\frac{1}{(m-1)x^{m-1}}+\frac{1}{2x^m}+\overset{\infty}{\mathrm{S}}_n(-1)^n.\lfloor\underline{m}_{\,2n-1,1}.\frac{\mathfrak{C}_{2n}}{\lfloor\underline{2n}.x^{m+2n-1}}$$

$$+\textit{const.}$$

[δ] $$\overset{x}{\mathrm{S}}_m\log_\epsilon m=\tfrac{1}{2}\log_\epsilon 2\pi+(x+\tfrac{1}{2})\log_\epsilon x-x$$

$$+\overset{\infty}{\mathrm{S}}_m(-1)^{m-1}.\frac{\mathfrak{C}_{2m}}{2m(2m-1)x^{2m-1}}.$$

(37.) The two following series will be found very useful in rendering the general term of a series integrable.

$$\phi(x)=\phi(u_1)+\left\{\frac{\phi(u_1)}{u_1-u_2}+\frac{\phi(u_2)}{u_2-u_1}\right\}(x-u_1)$$

$$+\left\{\frac{\phi(u_1)}{(u_1-u_2)(u_1-u_3)}+\frac{\phi(u_2)}{(u_2-u_1)(u_2-u_3)}\right.$$

$$\left.+\frac{\phi(u_3)}{(u_3-u_1)(u_3-u_2)}\right\}(x-u_1)(x-u_2)+\&c.$$

$$+\left\{\frac{\phi(u_1)}{(u_1-u_2)(u_1-u_3)\dots(u_1-u_n)}\right.$$

$$+\frac{\phi(u_2)}{(u_2-u_1)(u_2-u_3)\dots(u_2-u_n)}+\&c.$$

$$\left.+\frac{\phi(u_n)}{(u_n-u_1)(u_n-u_2)\dots(u_n-u_{n-1})}\right\}(x-u_1)(x-u_2)\dots(x-u_{n-1}).$$

Let $u_x=hx+k$, $S_0=1$,

$S_1=a_1+a_2+\&c.+a_n$, $S_2=a_1a_2+a_1a_3+a_2a_3+\&c.$

&c. = &c. $S_n=a_1a_2\dots a_n$; then

$$u_{x+a_1}.u_{x+a_2}\dots u_{x+a_n}=h^n.S_n$$

$$+\frac{h^{n-1}}{1}u_x\{\Delta 0.S_{n-1}-\Delta 0^2.S_{n-2}+\Delta 0^3.S_{n-3}-\&c.\}$$

$$+\frac{h^{n-2}}{1.2}u_x.u_{x+1}\{\Delta^2 0^2.S_{n-2}-\Delta^2 0^3.S_{n-3}+\&c.\}$$

$$+\frac{h^{n-3}}{1.2.3}.u_x.u_{x+1}.u_{x+2}\{\Delta^3 0^3.S_{n-3}-\&c.\}+\&c.$$ [α]

(*Herschel*, *Ex.* 165, 24.)

[α] $\overset{n}{\mathrm{P}}_m(u_{x+a_m})$

$$=\overset{n+1}{\mathrm{S}}_m\frac{h^{n-m+1}}{\lfloor m-1}.\overset{m-1}{\mathrm{P}}_r(u_{x+r-1}).\overset{n+1}{\mathrm{S}}_s(-1)^{m+s}.\Delta^{m-1}0^{s-1}.\overset{n-s+1,n}{\mathrm{C}}_t(a_t),$$

$$=h^n.\overset{n}{\mathrm{P}}_r(a_r)+\overset{n}{\mathrm{S}}_m\frac{h^{n-m}}{\lfloor m}.\overset{m}{\mathrm{P}}_r(u_{x+r-1}).\overset{n-m+1}{\mathrm{S}}_s(-1)^{s-1}.\Delta^m 0^{m+s-1}.\overset{n-m-s+1,n}{\mathrm{C}}_t(a_t).$$

(38.) *Recurring series.* The general form of a recurring series is

$$0 = au_{x+n} + a_1 u_{x+n-1} + \text{\&c.} + a_n u_x, \qquad (1)$$

by the integration of which the general term may be obtained, and thence the sum of the series.

By the following method the sum of a series may be obtained from its equation, without knowing the general term: for x write $x+1$, and substitute

$$u_{x+n-m} + \Delta_x u_{x+n-m} \text{ for } u_{x+n-m+1},$$

then (1) becomes

$$0 = a\Delta_x u_{x+n} + (a+a_1)\,\Delta_x u_{x+n-1} + \text{\&c.}$$
$$+ (a + a_1 + \text{\&c.} + a_{n-1})\,\Delta_x u_{x+1} + (a + a_1 + \text{\&c.} + a_n) u_{x+1},$$
$$\therefore \ \Sigma_x u_{x+1} = const.$$
$$- \frac{au_{x+n} + (a+a_1)u_{x+n-1} + \text{\&c.} + (a + a_1 + \text{\&c.} + a_{n-1})u_{x+1}}{a + a_1 + \text{\&c.} + a_n}$$

(*Tr. L.* App. 390—5.)

(39.) *Application of the integral calculus to series.*

The sum either of an infinite or a limited number of terms of many series may be represented by a definite integral: the following methods are applicable to numerous classes of series, consisting of ascending powers of some quantity, the coefficients of which are composed of arithmetical factorials. Let the series be

$$s = at^m + (a+b)t^{m+n} + (a+2b)t^{m+2n} + \text{\&c.}$$
$$+ \{a + (x-1)b\}\,t^{m+(x-1)n}:$$

multiplying by $\frac{n}{b}t^{\frac{a}{b}n-m-1}$, we obtain

$$\frac{n}{b}\int_t s t^{\frac{a}{b}n-m-1} = \frac{t^{\frac{a}{b}n} - t^{(\frac{a}{b}+x)n}}{1-t^n},$$

by differentiating which, the value of s may be obtained.

If $s = (a+b)(c+e)t^m + (2a+b)(2c+e)t^{m+n} + \&c.$
$+ (ax+b)(cx+e)t^{m+(x-1)n}$

then multiplying by $\frac{n}{c} t^{n\left(\frac{e}{c}+1\right)-m-1}$, and integrating, and again multiplying the result by $\frac{n}{a} t^{n\left(\frac{b}{a}-\frac{e}{c}\right)-1}$, and integrating, we have

$$\frac{n^2}{ac}\int_x t^{n\left(\frac{b}{a}-\frac{e}{c}\right)-1}\int_x s t^{n\left(\frac{e}{c}+1\right)-m-1} = t^{\frac{n(a+b)}{a}} \cdot \frac{1-t^{nx}}{1-t^n},$$

from which s may be obtained, after two differentiations.

The same method may be applied to the series of which the x^{th} term is

$$(a_1x+b_1)(a_2x+b_2)\dots(a_rx+b_r)\,t^{m+(x-1)n}.$$

Let $s = \dfrac{t^m}{a+b} + \dfrac{t^{m+n}}{2a+b} + \&c. + \dfrac{t^{m+(x-1)n}}{ax+b}$,

then multiplying by $\frac{a}{n} t^{\left(\frac{b}{a}+1\right)n-m}$, and differentiating, we obtain a series from which may be deduced

$$s = \frac{n}{a} t^{m-n\left(\frac{b}{a}+1\right)} \int_t t^{\left(\frac{b}{a}+1\right)n-1} \cdot \frac{1-t^{nx}}{1-t^n}.$$

Let $s = \dfrac{t}{(a+b)(c+e)} + \dfrac{t^2}{(2a+b)(2c+e)} + \&c. + \dfrac{t^x}{(ax+b)(cx+e)}$,

then by a similar process we obtain

$$s = \frac{1}{(bc-ae)t^{\frac{b}{a}}}\left\{t^{\frac{b}{a}-\frac{e}{c}}\int_t t^{\frac{e}{c}} \cdot \frac{1-t^x}{1-t} - \int_t t^{\frac{b}{a}} \cdot \frac{1-t^x}{1-t}\right\}.$$

If the x^{th} term is $\dfrac{t^x}{(ax+b)(cx+e)(fx+g)}$, then

$$s = \frac{1}{acft^{\frac{b}{a}}}\int_t t^{\frac{b}{a}-\frac{e}{c}-1}\int_t t^{\frac{e}{c}-\frac{g}{f}-1}\int_t t^{\frac{g}{f}} \cdot \frac{1-t^x}{1-t}.$$

If the x^{th} term is $\dfrac{t^x}{(ax+b)^m}$, then

$$s=\frac{1}{1.2\ldots(m-1)a^m.t^{\frac{b}{a}}}\Big\{(\log_\epsilon t)^{m-1}\int_t t^{\frac{b}{a}}.\frac{1-t^x}{1-t}$$

$$-\frac{m-1}{1}(\log_\epsilon t)^{m-2}\int_t t^{\frac{b}{a}}.\log_\epsilon t.\frac{1-t^x}{1-t}$$

$$+\frac{(m-1)(m-2)}{1\ .\ 2}(\log_\epsilon t)^{m-3}\int_t t^{\frac{b}{a}}.(\log_\epsilon t)^2.\frac{1-t^x}{1-t}-\&\text{c.}\Big\}.\ ^{a}$$

$$\frac{t}{1^n}\pm\frac{t^2}{2^n}+\frac{t^3}{3^n}\pm\&\text{c.}=\left(\int_{v=\log_\epsilon t}^{n}-\int_{v=-\infty}^{n}\right)\frac{\epsilon^v}{1\pm\epsilon^v},$$

where $v=\log_\epsilon t$.

Let $s=\dfrac{a+b}{c+e}t+\dfrac{2a+b}{2c+e}t^2+\&\text{c.}+\dfrac{ax+b}{cx+e}t^x$;

multiplying this by $ct^{\frac{e}{c}}$, and differentiating, and then multiplying the result by $\frac{1}{a}t^{\frac{b}{a}-\frac{e}{c}}$, and integrating, we obtain

$$s=\frac{a}{ct^{\frac{e}{c}}}\int_t\left\{t^{\frac{e}{c}-\frac{b}{a}}d_t t^{\frac{b}{a}-1}.\frac{1-t^x}{1-t}\right\}.$$

Let $s=(a+b)t+(a+b)(2a+b)t^2+\&\text{c.}$
$+(a+b)(2a+b)\ldots(ax+b)t^x$,

then multiplying by $\frac{1}{a}t^{\frac{b}{a}-1}$, we obtain the equation

$$at^2d_ts+\{(a+b)t-1\}s=(a+b)\ldots(\overline{x+1}.a+b)t^{x+1}-(a+b)t.$$

If the x^{th} term is $\dfrac{(a+b)(2a+b)\ldots(ax+b)}{(c+e)(2c+e)\ldots(cx+e)}t^x$, then multiplying by $\frac{1}{a}t^{\frac{b}{a}-1}$, and integrating, and then multiplying the

---

[a] $\overset{s}{\text{S}}_r\dfrac{t^r}{(ar+b)^m}=\dfrac{1}{a^m.t^{\frac{b}{a}}}\overset{m}{\text{S}}_n(-1)^{n-1}\dfrac{(\log_\epsilon t)^{m-n}}{\underline{\lvert m-n}}\displaystyle\int_t t^{\frac{b}{a}}.\dfrac{(\log_\epsilon t)^{n-1}}{\underline{\lvert n-1}}.\dfrac{1-t^x}{1-t}.$

quantity thus obtained by $ct^{\frac{e}{c}-\frac{b}{a}}$, and differentiating, the result is

$$\frac{c}{at^{\frac{e}{c}}}d_t\left\{t^{\frac{e}{c}-\frac{b}{a}}\int_t s t^{\frac{b}{a}-1}\right\}-1=s-\frac{(a+b)\dots(ax+b)}{(c+e)\dots(cx+e)}t^x.$$

(*Tr. L.* App. 412—5; *L. C. D.* 1140—8.)

THEORY OF GENERATING FUNCTIONS.

(40.) Let the indefinite series

$$\&\text{c.} + u_{-m}t^{-m} + \&\text{c.} + u_{-1}t^{-1} + u + u_1 t + \&\text{c.}$$
$$+ u_{x-1}t^{x-1} + u_x t^x + \&\text{c.}\ ^{\alpha}$$

be represented by $\phi(t)$: this is the ***generating function*** of u_x.

The generating function of u_{x+n} is $t^{-n}.\phi(t)$,

. . . . $\Delta_x u_x$ ... $\left(\frac{1}{t}-1\right)\phi(t)$,

. . . . $\Delta_x^n u_x$ ... $\left(\frac{1}{t}-1\right)^n\phi(t)$,

. . . . $\Sigma_x^n u_x$ ... $\left(\frac{1}{t}-1\right)^{-n}\phi(t)$;

from the development of $t^{-n}\phi(t)$, when put under the form $\left\{1+\left(\frac{1}{t}-1\right)\right\}^n\phi(t)$, and of $\left(\frac{1}{t}-1\right)^n\phi(t)$, the values of u_{x+n}, and $\Delta_x^n u_x$ in *Art.* 2 may be obtained.

(41.) Let ∇u_x be used to represent the series

$$au_x + a_1 u_{x+1} + a_2 u_{x+2} + \&\text{c.} + a_n u_{x+n},\ ^{\beta}$$

the generating function of which is

$$\left(a+\frac{a_1}{t}+\frac{a_2}{t^2}+\&\text{c.}+\frac{a_n}{t^n}\right)\phi(t):$$

[α] $\phi(t)=\overset{-\infty,\infty}{\mathrm{S}_x} u_x t^x$.

[β] $\nabla u_x = \overset{n+1}{\mathrm{S}_m} a_{m-1} u_{x+m-1}$.

and let $\nabla^2 u_x = a\nabla u_x + a_1 \nabla u_{x+1} + \&c. + a_n \nabla u_{x+n}$,
&c. = &c.: then the generating function of $\nabla^m u_x$ is

$$\left(a + \frac{a_1}{t} + \frac{a_2}{t^2} + \&c. + \frac{a_n}{t^n}\right)^m \phi(t),$$

and that of $\Delta^r \nabla^m u_{x\pm p}$ is

$$\left(a + \frac{a_1}{t} + \&c. + \frac{a_n}{t^n}\right)^m . \left(\frac{1}{t} - 1\right)^r . t^{\mp p} . \phi(t).$$

If we assume $x = t\left(\frac{1}{t} - 1\right)^2$, and develope $\frac{1}{t^n}$ in a series of powers of x, we may obtain the formulæ in *Art.* 8.

(42.) A very general formula for interpolation may be obtained in the following manner. Let

$$x = a + \frac{a_1}{t} + \frac{a_2}{t^2} + \&c. + \frac{a_m}{t^m}; \text{ [γ]}$$

it is required to obtain an expression for t^{-n}, arranged according to powers of x, and containing no higher negative powers of t than t^{-m}. Multiplying the numerator and denominator of the fraction $\frac{1}{1 - vt^{-1}}$ by

$$(a - x)v^m + a_1 v^{m-1} + \&c.\ a_{m-1} v + a_m,$$

and substituting for x its value in the numerator, we obtain

$$\frac{a_1 v^{m-1} + a_2 v^{m-2} + \&c. + a_m + \frac{v}{t}(a_2 v^{m-2} + \&c. + a_{m-1} v + a_m) + \frac{v^2}{t^2}(a_3 v^{m-3} + \&c. + a_{m-1} v + a_m) + \&c. + a_m \frac{v^{m-1}}{t^{m-1}}}{av^m + a_1 v^{m-1} + \&c. + a_{m-1} v + a_m - xx^m}: \text{ [δ]}$$

[γ] $x = \overset{m+1}{S}_n \frac{a_{n-1}}{t^{n-1}}$.

[δ] $\frac{\overset{m}{S}_n (v . t^{-1})^{m-n} . \overset{n}{S}_r a_{m-r+1} v^{n-1}}{\overset{m+1}{S}_n a_{n-1} v^{m-n+1} - xv^m}$.

let $$av^m + a_1 v^{m-1} + \&c. + a_{m-1}v + a_m = V,\ ^{\alpha}$$
and let $Z_{s-1,r}$ represent the coefficient of v^r in the expansion of V^s, then the value of $\frac{1}{t^n}$ which is the coefficient of v^n in the development of the above fraction is

$$\begin{aligned}\frac{1}{t^n} = {} & a_1 Z_{0,n-m+1} + a_2 Z_{0,n-m+2} + \&c. + a_m Z_{0,n} \\ & + x(a_1 Z_{1,n-2m+1} + a_2 Z_{1,n-2m+2} + \&c. + a_m Z_{1,n-m}) \\ & + x^2(a_1 Z_{2,n-3m+1} + a_2 Z_{2,n-3m+2} + \&c. + a_m Z_{2,n-2m}) \\ & + \&c.\end{aligned}$$

$$+ \frac{1}{t}\left\{\begin{aligned} & a_2 Z_{0,n-m+1} + a_3 Z_{0,n-m+2} + \&c. + a_m Z_{0,n-1} \\ & + Z(a_2 Z_{1,n-2m+1} + a_3 Z_{1,n-2m+2} + \&c. + a_m Z_{1,n-m-1}) \\ & + \&c.\end{aligned}\right.$$

$$+ \frac{1}{t^2}\left\{\begin{aligned} & a_3 Z_{0,n-m+1} + \&c. + a_m Z_{0,n-2} \\ & + Z(a_3 Z_{1,n-2m+1} + \&c. + a_m Z_{1,n-m-2}) \\ & + \&c.\end{aligned}\right.$$

$$+ \&c.$$

$$+ \frac{1}{t^{m-1}}\{a_m Z_{0,n-m+1} + x a_m Z_{1,n-2m+1} + \&c.\};\ ^{\beta}$$

multiplying by $\phi(t)$, and passing from the generating functions to the quantities which they represent, we have

$$\begin{aligned}u_{x+n} = {} & u_x\{a_1 Z_{0,n-m+1} + a_2 Z_{0,n-m+2} + \&c. + a_m Z_{0,n}\} \\ & + \nabla u_x\{a_1 Z_{1,n-2m+1} + a_2 Z_{1,n-2m+2} + \&c. + a_m Z_{1,n-m}\} \\ & + \nabla^2 u_x\{a_1 Z_{2,n-3m+1} + a_2 Z_{2,n-3m+2} + \&c. + a_m Z_{2,n-2m}\} \\ & + \&c.\end{aligned}$$

[α] $V = \overset{m+1}{\mathrm{S}}_n a_{n-1} v^{m-n+1}.$

[β] $\frac{1}{t^n} = \overset{m}{\mathrm{S}}_p \frac{1}{t^{p-1}} \cdot \overset{\infty}{\mathrm{S}}_r x^{r-1} \cdot \overset{m-p+1}{\mathrm{S}}_s a_{p+s-1} \varpi^{n-rm+s} b^r$, $\left(b_{q-1} = a_{m-q+1}, \left\{\begin{smallmatrix} q=1, \\ q=m+1\end{smallmatrix}\right\}\right)$

$$\begin{aligned}
&+ u_{x+1}\ \{a_2 Z_{0,n-m+1} + a_3 Z_{0,n-m+2} + \&c. + a_m Z_{0,n-1}\} \\
&+ \nabla u_{x+1}\{a_2 Z_{1,n-2m+1} + a_3 Z_{1,n-2m+2} + \&c. + a_m Z_{1,n-m-1}\} \\
&+ \&c. \\
&+ u_{x+m-1} a_m Z_{0,n-m+1} + \nabla u_{x+m-1} a_m Z_{1,n-2m+1} + \&c.
\end{aligned}$$ [γ]

If a value can be assigned to q that will render $\nabla^q u_x = 0$, then the above value of u_{x+n} will consist of a finite number of terms.

If $\nabla u_x = 0$, the above series will afford a complete solution of a linear equation of differences of the n^{th} order: in that case the quantities u, u_1, &c. u_{n-1}, represent the arbitrary constants.

(*Laplace*, *Theor. Analyt. des Prob.* Pt. 1; *L. C. D.* 1109—33; *Tr. L.* App. 346—9.)

[γ] $u_{x+n} = \overset{m}{\mathrm{S}}_p \overset{\infty}{\mathrm{S}}_r \nabla^{r-1} u_{x+p-1} \overset{m-p+1}{\mathrm{S}}_s a_{p+s-1} \varpi^{n-rm+s} b^r.$

◆

FUNCTIONAL EQUATIONS.

REDUCTION OF FUNCTIONAL EQUATIONS TO EQUATIONS OF DIFFERENCES.

(1.) This reduction can in very few cases be effected, so that the result may be an integrable equation of differences: in the equation

$$0=F\{x,\psi\{a_1(x)\},\psi\{a_2(x)\}\}, \qquad (1)$$

in which F, a_1, and a_2, are known functions* assume

$$a_1(x)=\phi(z), \qquad a_2(x)=\phi(z+1),$$

in which z is a function of x. Eliminating x from these equations we obtain

$$0=\phi(z+1)-a_2\{a_1^{-1}\{\phi(z)\}\},$$

from which the function ϕ may be determined: substituting for x, $a_1(x)$, $a_2(x)$, their values, (1) becomes

$$0=F\{a_1^{-1}\{\phi(z)\},\psi\{\phi(z)\},\psi\{\phi(z+1)\}\};$$

let $\chi(z)$ be the value of $\psi\{\phi(z)\}$ obtained from this equation, then

$$\psi(z)=\chi\{\phi^{-1}(z)\}.$$

(2.) The same method of solution cannot be applied to the equation

$$0=F\{x,\psi\{a_1(x)\},\psi\{a_2(x)\},..\psi\{a_{n+1}(x)\}\}, \qquad (1)$$

a solution may however be obtained by the following method.

---

* In the following pages $F, f, \alpha, \beta, \gamma$, will, for the sake of distinction, be used as the characteristics of known functions; ξ, ϕ, χ, ψ, being unknown or arbitrary functions.

Let z_1, z_2, &c. z_n, be functions of x such that

$$\begin{aligned} a_1(x) &= \phi(z_1, z_2, \dots z_n), \\ a_2(x) &= \phi(z_1 - 1, z_2, \dots z_n), \\ &- - - - - - - - - - - - - \\ a_{n+1}(x) &= \phi(z_1, z_2 \dots z_n - 1): \end{aligned} \qquad (2)$$

eliminating x from these equations we obtain

$$\begin{aligned} a_1^{-1}\{\phi(z_1, z_2, \dots z_n)\} &= a_2^{-1}\{\phi(z_1 - 1, z_2, \dots z_n)\}, \\ \dots\dots\dots\dots\dots &= a_3^{-1}\{\phi(z_1, z_2 - 1, \dots z_n)\}, \\ - - - - - - - - - - &- - - - - - - - - - - - - \\ \dots\dots\dots\dots\dots &= a_{n+1}^{-1}\{\phi(z_1, z_2, \dots z_n - 1)\}; \end{aligned} \qquad (3)$$

these are equations of partial differences of the first order, the integrals of which will be

$$\begin{aligned} \phi(z_1, z_2, \dots z_n) &= f_1\{z_1, z_2, \dots z_n, \chi_1(\cos 2\pi z_1, z_2, \dots z_n)\}, \\ - - - - - - - - - - &- - - - - - - - - - - - - - - \\ \dots\dots\dots\dots &= f_n\{z_1, z_2, \dots z_n, \chi_n(z_1, z_2, \dots \cos \pi z_n)\}, \end{aligned} \qquad (4)$$

in which χ_1, &c. are arbitrary functions introduced by integration. Let $a_1^{-1}\{\phi(z_1, z_2, \dots z_n)\}$ be substituted for x in (1), which then becomes

$$\begin{aligned} 0 = F\Big\{ a_1^{-1}\{\phi(z_1, \dots z_n)\}, \psi\{\phi(z_1, \dots z_n)\}, \psi\{\phi(z_1 - 1, \dots z_n)\}, \\ \dots\dots \psi\{\phi(z_1, \dots z_n - 1)\}\Big\} \end{aligned}$$

an equation of partial differences from the solution of which we may obtain

$$\psi\{\phi(z_1, \dots z_n)\} = \psi\{a_1(x)\} = F_1(z_1, \dots z_n, Z_1, Z_2 \text{ \&c.})$$

in which Z_1, Z_2, &c. represent the arbitrary functions χ_1, &c. in (4).

If the values of z_1, z_2, &c. z_n, obtained by elimination from (4) be substituted in this equation, $F_1(z_1$ &c.) will become a function of $a_1(x)$, and the function ψ may be determined.

(3.) The general equation containing two variables

$$0 = F\big\{x, y, \psi\{a_1(x), \beta_1(y)\}, \dots \psi\{a_{n+1}(x), \beta_{n+1}(y)\}\big\}$$

may be treated in a similar manner by assuming two systems of equations,

$$\begin{array}{ll}
\alpha_1(z) = \phi_1({}^1z_1, {}^1z_2, \ldots {}^1z_n), & \beta_1(y) = \phi_2({}^2z_1, {}^2z_2, \ldots {}^2z_n), \\
\alpha_2(x) = \phi_1({}^1z_1 - 1, {}^1z_2 \ldots {}^1z_n), & \beta_2(y) = \phi_2({}^2z_1 - 1, {}^2z_2, \ldots {}^2z_n), \\
- - - - - - - - - - - - - - - & - - - - - - - - - - - - - - - \\
\alpha_{n+1}(x) = \phi_1({}^1z_1, {}^1z_2, \ldots {}^1z_n - 1)\,; & \beta_{n+1}(y) = \phi_2({}^2z_1, {}^2z_2, \ldots {}^2z_n - 1)\,;
\end{array}$$

which may be solved in the same manner as the equations (2).

If $\alpha(x) = \dfrac{a + bx}{c + ex}$, $a.x^b$, or $a.c^x$, or any combination of these quantities, the equations of differences involved in the above method will all be integrable.

(*Mem. Analyt. Soc.* 1813, pp. 96—114.)

GENERAL SOLUTION OF FUNCTIONAL EQUATIONS, BY MEANS OF PARTICULAR SOLUTIONS.

(4.) *Functional equations of the first order.*

Let $f(x) = f\{\alpha(x)\}$ be a particular solution of the equation

$$\psi(x) = \psi\{\alpha(x)\},$$

then the general solution will be

$$\psi(x) = \phi\{f(x)\},$$

in which ϕ is an arbitrary function.

If the given particular solution is

$$f(x) = f\{\alpha^2(x)\},$$

the general solution will be

$$\psi(x) = \phi\,\{\overline{f(x), f\{\alpha(x)\}}\}:$$

the function ϕ being arbitrary. The line denotes that the function is symmetrical with regard to those quantities, over which it is placed.

Let f be a particular solution of the equation

$$\psi(x) = \beta(x) \cdot \psi\{a(x)\}, \qquad (1)$$

and f_1 of the equation $\psi(x) = \psi\{a(x)\}$, then the general solution of (1) will be

$$\psi(x) = f(x) \cdot \phi\{f_1(x)\}.$$

If f_2 is a particular solution of the equation

$$\psi(x) = \psi\{a^2(x)\},$$

then the general solution of (1) will be

$$\psi(x) = f(x) \cdot \phi\{\overline{f_2(x)},\ \overline{f_2\{a(x)\}}\}.$$

Let $f(x)$ be a particular solution of the equation

$$\psi(x) = \beta(x) \cdot \psi\{a(x)\} + \gamma(x),$$

then by assuming $\psi(x) = f(x) + \phi(x)$, this may be reduced to the preceding case.

By a similar assumption, the equation

$$0 = \psi(x) + \beta_1(x) \cdot \psi\{a_1(x)\} + \&c. + \beta_n(x) \cdot \psi\{a_n(x)\} + \gamma(x)$$

may be reduced to

$$0 = \psi(x) + \beta_1(x) \cdot \psi\{a_1(x)\} + \&c. + \beta_n(x) \cdot \psi\{a_n(x)\}.$$

(5.) Given $\psi(x) = \psi\{a_1(x)\} = \&c. = \psi\{a_n(x)\}$, or as the same system of equations may be more conveniently expressed,

$$\psi(x) = \psi\{a_m(x)\}, \left\{\begin{smallmatrix} m=1 \\ m=n \end{smallmatrix}\right\},$$

to determine the function ψ: let f_1, f_2, &c. f_n be so assumed as successively to satisfy the system of equations

$$f_m \underset{m}{\{} f_{m-1} \cdots \underset{2}{\{} f_1 \underset{1}{\{} x \underset{1}{\}} \underset{2}{\}} \cdots \underset{m}{\}} = f_m \underset{m}{\{} f_{m-1} \cdots \underset{2}{\{} f_1 \underset{1}{\{} a_m(x) \underset{1}{\}} \underset{2}{\}} \cdots \underset{m}{\}}, \left\{\begin{smallmatrix} m=1 \\ m=n \end{smallmatrix}\right\},$$

then

$$\psi(x) = \phi \underset{n}{\{} f_n \underset{n-1}{\{} \cdots \underset{1}{\{} f_1(x) \underset{1}{\}} \cdots \underset{n}{\}},$$

in which ϕ is arbitrary, will be the general solution.

If it be required to determine ψ from the conditions

$$\psi\{a_m(x)\} = \psi\{\beta_m(x)\}, \left\{\begin{smallmatrix} m=1 \\ m=n \end{smallmatrix}\right\},$$

let f_1, f_2, &c. be so assumed as to satisfy the equations

$$f_m \underset{m}{\{} \cdots f_1 \underset{1}{\{} a_1 \underset{1}{(} \cdots a_m \underset{m}{(} x \underset{m}{)} \cdots \underset{1}{)} \underset{1}{\}} \cdots \underset{m}{\}} = f_m \underset{m}{\{} \cdots f_1 \underset{1}{\{} a_1 \underset{1}{(} \cdots a_{m-1} \underset{m-1}{(} \beta_m \underset{m}{(} x \underset{m}{)} \cdots \underset{m}{\}}, \ \left\{ {m=1 \atop m=n} \right\},$$

then $$\psi(x) = \phi \underset{n}{\{} f_n \cdots \underset{1}{\{} f_1 \{ a_1 \underset{1}{(} \cdots a_n \underset{n}{(} x \underset{n}{)} \cdots \underset{n}{\}}.$$

If a general solution of the equation

$$0 = F \{ x, \psi(x), \psi\{a_1(x)\}, \ldots \psi\{a_n(x)\} \},$$

be required, having given $\psi(x) = f(x, a_1, a_2, \ldots a_r)$, a particular solution containing r arbitrary constants; let ϕ be determined such that

$$\phi(x) = \phi\{a_m(x)\}; \ \left\{ {m=1 \atop m=n} \right\},$$

then the general solution will be

$$\psi(x) = f\{x, \phi_1(x), \ldots \phi_r(x)\},$$

in which ϕ_1, &c. are particular values of ϕ.

(6.) *Functional equations of the second and higher orders.*

The equation $\psi^2(x) = x$ will be satisfied by any value of ψ obtained from the equation

$$\phi\{\overline{x, \psi(x)}\} = 0:$$

let f be a particular value of ψ in this equation, then the general solution will be

$$\psi(x) = \phi^{-1}\{f\{\phi(x)\}\},$$

ϕ being arbitrary; it is however necessary to observe that those values only of ϕ^{-1} should be taken, which satisfy the equation

$$\phi^{-1}\{\phi(x)\} = x.$$

If f is a particular value of ψ in the equation

$$\psi^n(x) = x,$$

then the general solution will be

$$\psi(x) = \phi^{-1}\{f\{\phi(x)\}\}.$$

Let the proposed equation be $\psi^2(x) = \alpha(x)$; assume

$$\psi(x) = \phi^{-1}\{f\{\phi(x)\}\},$$

then $$f^2\{\phi(x)\} = \phi\{\alpha(x)\},$$

which is an equation of the first order with respect to ϕ, and may be solved by preceding methods.

By the same substitution the equation $\psi^n(x) = \alpha(x)$ may be reduced to $f^n\{\phi(x)\} = \phi\{\alpha(x)\}$.

Given $$\psi\{\alpha\{x, \psi(x)\}\} = x:$$

assume $$\psi(x) = \phi^{-1}\{f\{\phi(x)\}\},$$

then $$\alpha\{\phi^{-1}(x), \phi^{-1}\{f(x)\}\} = \phi^{-1}\{f^{-1}(x)\},$$

which is an equation of the first order with respect to ϕ^{-1}.

The equation

$$0 = F\{x, \psi(x), \psi^2\{\alpha_1(x)\}, \ldots \psi^n\{\alpha_{n-1}(x)\}\}$$

may be reduced to an equation of the first order

$$0 = F\{\phi^{-1}(y), \phi^{-1}\{f(y)\}, \ldots \phi^{-1}\{f^n(y)\}\}, \qquad (a)$$

by assuming $\psi(x) = \phi^{-1}\{f\{\phi(x)\}\}$, and putting y for $\phi(x)$; f being a particular value of ψ, and ϕ such that

$$\phi(x) = \phi\{\alpha_m(x)\}, \ \left\{\begin{smallmatrix} m=1 \\ m=n-1 \end{smallmatrix}\right\}.$$

The equation $0 = F\{x, \psi(x), \psi^2(x), \ldots \psi^n(x)\}$ may be reduced to the form (a) by assuming

$$\psi(x) = \phi^{-1}\{f\{\phi(x)\}\},$$

in which ϕ is arbitrary, and f a particular value of ψ.

(*Babbage, Phil. Trans.* 1815.)

(7.) A particular solution of the equation

$$0 = F\{x, \psi(x), \psi\{\alpha(x)\}, \psi\{\alpha^2(x)\}, \ldots \psi\{\alpha^n(x)\}\},$$

α being such that $\alpha^{n+1}(x)=x$, may be thus obtained: by substituting successively $\alpha(x)$, $\alpha^2(x)$, &c. $\alpha^n(x)$ for x, we obtain

$$0=F\{\alpha^{r-1}(x),\ \psi\{\alpha^{r-1}(x)\},\ \psi\{\alpha^r(x)\},\ldots\psi\{\alpha^n(x)\},\ \psi\{\alpha(x)\},$$
$$\ldots\psi\{\alpha^{r-2}(x)\}\},\ \left\{{r=1 \atop r=n+1}\right\},$$

from which $n+1$ equations the n quantities $\psi\{\alpha(x)\}$, &c. $\psi\{\alpha^n(x)\}$, may be eliminated, and the resulting equation will be

$$0=f\{x,\ \alpha(x),\ \alpha^2(x),\ldots\alpha^n(x),\ \psi(x)\},$$

from which the function ψ may be found.

This method is inapplicable if the proposed equation either does not contain x; or is homogeneous with respect to $\psi(x)$, $\psi\{\alpha(x)\}$, &c.; or can be put under the form

$$F_1\{\psi(x),\ \psi\{\alpha(x)\},\ldots\psi\{\alpha^n(x)\}\}=f(x),$$

unless $$f(x)=f_1\{x,\ \alpha(x),\ \alpha^2(x),\ldots\alpha^n(x)\}.$$

(*Babbage, Phil. Trans.* 1816, pp. 229—34.)

(8.) *Functional equations containing two unknown quantities.*

Let the proposed equation be

$$\psi(x,y)=\psi\{\alpha(x),\ \beta(y)\},$$

and let f_1, f_2, be functions which satisfy the equations

$$f_1(x)=f_1\{\alpha(x)\},\qquad f_2(y)=f_2\{\beta(y)\},$$

then the general solution will be

$$\psi(x,y)=\phi\{f_1(x),\ f_2(y)\}.$$

If f is a particular value of ψ, the general solution may be put under the form

$$\psi(x,y)=\phi\{f\{\alpha(x),\ \beta(y)\}\}.$$

The general solution of the equation

$$\psi(x,y)=\psi\{\alpha(x,y),\ \beta(x,y)\} \qquad (1)$$

is $$\psi(x,y)=\psi\{f_1(x,y),\ f_2(x,y)\},$$

in which f_1 and f_2 are particular values of ψ, and ϕ arbitrary.

The general solution of the equation

$$\psi(x,y)=\gamma(x,y)\cdot\psi\{\alpha(x,y),\beta(x,y)\} \qquad (2)$$

is $$\psi(x,y)=f(x,y)\cdot\psi\{f_1(x,y),f_2(x,y)\},$$

in which f is a particular value of ψ in the proposed equation, and f_1; f_2, particular solutions of (1).

The equation

$$\psi(x,y)=\gamma_1(x,y)\cdot\psi\{\alpha(x,y),\beta(x,y)\}+\gamma_2(x,y)$$

may be reduced to (2) by assuming

$$\psi(x,y)=f(x,y)+\phi(x,y),$$

in which f is a particular value of ψ.

(9.) *Functional equations of the second and superior orders, containing two variables.* The equations

$$\psi^{2,1}(x,y)=x, \qquad \psi^{2,1}(x,y)=\alpha(x,y),$$

may be treated in the same manner as the equations

$$\psi^2(x)=x, \qquad \psi^2(x)=\alpha(x),$$

arbitrary functions of y being substituted for the arbitrary constants contained in the general solutions of the latter equations.

A general solution of the equation $\psi^2(x,y)$* $=a$ is

$$\psi(x,y)=\frac{a\chi\left\{(x-y)\phi(x,y),\xi\left(\frac{x}{y}\right)\right\}}{\chi\{0,\xi(1)\}},$$

in which χ, ϕ, and ξ, are arbitrary functions.

If ψ is a homogeneous function of n dimensions, then

$$\psi^{r+1}(x,y)=\overline{\psi(x,y)}\big|^{n^r}\cdot\overline{\psi(1,1)}\big|^{\frac{1-n^r}{1-n}}.$$

---

* $\psi^2(x,y)$ has the same signification with $\psi^{\overline{2,2}}(x,y)$, as used by Mr. Babbage.

The general solution of the equation

$$\psi^2(x,y)=\gamma\{\psi(x,y)\}$$

is
$$\psi(x,y)=\gamma\left\{\frac{\phi(x,y)}{\chi(x,y)}\right\},$$

in which ϕ and χ are homogeneous functions of $n+1$, and n dimensions respectively, and such that $\phi(1,1)=\chi(1,1)$.

If the proposed equation is

$$\psi^n(x,y)=\gamma\{\psi(x,y)\},$$

the general solution is $\psi(x,y)=\xi\left\{\dfrac{\phi(x,y)}{\chi(x,y)}\right\}$,

ϕ and χ being as above, and ξ determined from the equation

$$\xi^{n-1}(z)=\gamma(z).$$

The same solution applies to the equation

$$\psi^n(x,y)=\gamma\{\psi^{n-r}(x,y)\},$$

ξ being in this case determined by the equation

$$\xi^r(z)=\gamma(z).$$

The equations
$$x.\psi^{1,2}(x,y)=y.\psi^{2,1}(x,y),$$
$$\psi^{2,1}(x,y).\psi^{1,2}(x,y)=xy,$$
$$\gamma\{x,\psi^{1,2}(x,y)\}=\gamma\{\psi^{2,1}(x,y),y\},$$

may be solved by assuming

$$\psi(x,y)=\phi^{-1}\{f\{\phi(x),\phi(y)\}\},$$

f being determined by the equations

$$f^{2,1}(x,y)=x, \qquad f^{1,2}(x,y)=y.$$

The equations
$$x.\psi^2(x,y)=a.\psi^{2,1}(x,y),$$
$$\gamma\{\psi^2(x,y),x\}=\gamma\{a,\psi^{2,1}(x,y)\},$$

may be solved by the same assumption; f being in the case determined by the equations

$$f^2(x,y)=\phi(a), \qquad f^{2,1}(x,y)=x.$$

(*Babbage*, *Phil. Trans.* 1816, pp. 184—222.)

(10.) Given the equation $\psi\{\alpha(x)\}=d_x\psi(x)$, in which $\alpha^2(x)=x$,

then $$\psi\{\alpha(x)\}=d_x d_{\alpha(x)}\psi(x)=d_x\frac{d_x\psi\{\alpha(x)\}}{d_x\alpha(x)}.$$

Given $\psi\{\alpha(x)\}=d_x^n\psi(x)$, and $\alpha^p(x)=x$;
by successively substituting $\alpha(x)$, $\alpha^2(x)$, &c. $\alpha^p x$, for x, we obtain

$$\psi(x)=d^n_{\alpha^{p-1}(x)}\,d^n_{\alpha^{p-2}(x)}\ldots d^n_{\alpha(x)}\,d^n_x\psi(x),$$

a partial differential equation from the solution of which ψ may be found.

The equation $0=$

$$F\{x,\psi(x),\psi\{\alpha(x)\},\ldots\psi\{\alpha^{p-1}(x)\},d_x^{m_1}\psi\{\alpha(x)\},\ldots d_x^{m_{p-1}}\psi\{\alpha(x)\}\}$$

in which $\alpha^p(x)=x$ may be solved in a similar manner by substituting successively $\alpha(x)$, &c. $\alpha^{p-1}(x)$, for x, and obtaining a series of equations from which all quantities except x, $\psi(x)$, and its differential coefficients may be eliminated.

(11.) Given $\psi(x,y)=d_x\psi\{x,\alpha(y)\}$, in which $\alpha^2(y)=y$,
then $\psi(x,y)=\epsilon^x.\phi_1\{\overline{y,\alpha(y)}\}+\epsilon^{-x}\{\alpha(y)-y\}.\phi_2\{\overline{y,\alpha(y)}\}$,
is the general solution, in which ϕ_1 and ϕ_2 are arbitrary functions.

If $\psi(x,y)=d_x\psi\{x,\alpha(y)\}$, and $\alpha^p(y)=y$, then

$$\psi(x,y)=d_x^p\psi(x,y),$$

from the solution of which equation ψ may be obtained.

Given $d_x\psi\{x,\beta(y)\}=d_y\psi\{\alpha(x),y\}$,
in which $\alpha^2(x)=x$, and $\beta^2(y)=y$; then ψ may be found from the equation $d_y\beta(y).d_x^2\psi(x,y)=d_x\alpha(x).d_y^2\psi(x,y)$.

The equation

$$0=F\{x,y,\psi(x,y),\ldots\psi\{\alpha^m(x),\beta^n(y)\},\ldots d_x^r d_y^s\psi\{\alpha^t(x),\beta^v(y)\}\}$$

in which $\alpha^p(x)=x$, and $\beta^q(y)=y$, may be solved in the same manner as the above equation of the same form, which contains only one unknown quantity.

(*Babbage, Phil. Trans.* 1816, pp. 235—52.)

APPENDIX.

The system of notation which has been adopted in the preceding pages, will have been found to vary in several particulars from that which has been long sanctioned and generally applied. It may therefore be expected that some sufficient reasons should be assigned for such a deviation from an established usage; and particularly as every change in the form of known symbols is calculated to interpose new difficulties to the pursuit of mathematical investigation.

The opinions on the subject of notation which have induced the changes alluded to, cannot be more forcibly expressed than in the words of two of the ablest Mathematicians of the present day:

"The great object of all notation is to convey to the mind, as speedily as possible, a complete idea of operations which are to be, or have been, executed; and since everything is to be exhibited to the eye, the more compact and condensed the symbols are, the more readily will they be caught, as it were, at a glance."*

"The symmetry of mathematical notation (which should ever be guarded with a jealousy commensurate to its vital importance) facilitates the translation of an expression into common language at any stage of an operation, disburdens the memory of all the load of previous steps, and at the same time affords it a considerable assistance in retaining the results."

---

* *Edinburgh Encyclopedia,* Art. *Notation.*

"To employ as many symbols of operation, and as few of quantity as possible is a precept which is now found to ensure elegance and brevity."*

In addition to these maxims it may be stated that those symbols will be found most convenient, which contain the greatest number of indications, and which, provided such indications are clearly and distinctly expressed, are calculated to afford the most effectual assistance to the memory.

The various branches of analysis have been discovered at different periods, and enriched by the labours of different individuals, and it may not therefore be unreasonably assumed, that the modes of notation which they have respectively introduced in their writings, although perhaps the best adapted to the then existing state of the science, should yet be susceptible of considerable improvement, in proportion as the mutual relations of distinct and hitherto unconnected branches of analysis are more clearly and accurately investigated.

The different systems of notation which have been adopted in the differential calculus appear to involve, to a certain extent, the different theories on which that calculus has been founded. Among these, the establishment of the mutual relations of certain functions by Lagrange is perhaps the simplest, and at the same time the most disencumbered of considerations foreign to the subject: although on the other hand the symbols by which those relations were originally expressed are liable to almost as many objections as any that have been devised.

The method of Lagrange may be thus stated: let $\phi(x)$ represent any function of x, and let x be supposed to receive an increment Dx, then the difference between $\phi(x)$ and $\phi(x+Dx)$ may always be developed in a series consisting of ascending powers of the quantity Dx multiplied by some functions of x and of any other quantities contained in $\phi(x)$.

---

* Preface to the *Memoirs of the Analytical Society*, Cambridge, 1813.

The coefficient of Dx in this development has been called the first differential coefficient of $\phi(x)$, and the coefficients of $\frac{(Dx)^2}{1.2}$, &c. $\frac{(Dx)^m}{1.2\ldots m}$, the second, &c. m^{th}, differential coefficients respectively: for these the symbols proposed by Lacroix have been adopted in the preceding pages.

On whatever hypothesis the differential calculus may have been established, the correctness of the results obtained by the application of the *inverse process*, or integral calculus, must necessarily depend on this one axiom; that two essentially different developments of the above-mentioned form cannot be obtained from the same function, or in other words, two different functions cannot have the same differential coefficient; or the same *differential*, or *fluxion*, according to the theory adopted. The object of the various artifices of the integral calculus may be stated to be the separation of any proposed quantity into a sum of other quantities having the form of previously ascertained differential coefficients, from which the corresponding integrals are known by an inductive process.

Considering the intimate connexion that exists between the differential calculus, and the calculus of finite differences, it is obviously desirable to render the analogy between their symbols as complete as possible: it is with this view that the previously adopted symbols d, $\int$, Δ, Σ, have been somewhat modified. A superior index has been used alike in all these cases to express the number of times that the operation indicated by the symbol is performed, and an inferior index to denote the quantity with respect to which that operation is effected.

As there appears a want of sufficient perspicuity in the ordinary application of the symbol Δ, the character D has been introduced to denote a difference of a quantity generally, Δ having been restricted to that particular case in which the difference of the variable is unity; the ordinary symbol is thus retained in that case in which it has been most frequently applied.

The method of denoting the powers of a quantity by positive or negative indices has long been universally adopted: the signification of indices has however more recently been considerably extended by their application to symbols of operation; denoting in this case a repetition of that operation, and when negative, the performance of the corresponding inverse process.

It is undoubtedly desirable to avoid as far as possible a multiplicity of arbitrary symbols; and with this view, the characters $\int$ and Σ might be deemed superfluous, as being identical with d^{-1}, and D^{-1} or Δ^{-1}; they may however be retained for the convenience of writing, more particularly as they involve no breach of analogy.

With regard to partial differential equations, the notation that has been adopted in the preceding pages, though it appears more in accordance with the general system, and with other parts of the calculus, than that ordinarily used, must nevertheless be acknowledged to be far from satisfactory. The nature of these equations, and their connexion with the general theory of functions is probably not yet sufficiently understood, to render them subservient to any general system of notation.

In denoting the powers of the trigonometrical functions, the notation of Arbogast, as $(\sin x)^m$, has been adopted. The index has been placed in the same situation by Gauss, Delambre, and others, but the parentheses have been omitted by these writers, without perhaps sufficient reason; for although the trigonometrical functions of the powers of any quantity have not hitherto been the subject of investigation, there does not appear any reason why they should not hereafter form the basis of important analytical relations, in which case the introduction of parentheses would be indispensable: and it appears to be a desirable object in the construction of symbols to render them applicable not only to existing circumstances, but to provide as far as possible for those cases which we have it in our power to contemplate. As the sine, cosine, &c. may be considered merely as functions, independently of all geometrical considerations, and as $\sin^{-1} x$, &c. have been used to represent the corresponding inverse functions, the more usual notation of $\sin^n x$ cannot longer be retained without an entire disregard of that uniformity of system, which it appears so desirable to establish.

The convenience of denoting the value which any function $\phi(u)$ assumes, when a is substituted for x in the quantity u, by $\phi_{x=a}(u)$ may be adduced as an instance of the advantage of indicating by symbols as many of the conditions of a problem as can be conveniently introduced. Examples of the use of this symbol will be seen in its application to Maclaurin's theorem, and to the decomposition of rational fractions. In expressing the value of a definite integral, the symbols of operation have been separated from those of quantity, merely for the convenience of writing.

Having thus briefly alluded to those deviations from generally acknowledged symbols which have been introduced in the *text*, an explanation will now be offered of a very compendious as well as comprehensive system of notation proposed by Mr. Jarrett, an account of which will be found in the third Volume of the Transactions of the Cambridge Philosophical Society.

As the preceding pages are designed to form a compendium of results for the use of the analyst, the various expressions to which this notation is applicable appear under their usual form in the *text*, lest by being put under an entirely new form, the volume might be rendered less useful to those who are not disposed to undertake the acquisition of an entirely new mathematical language, the real value of which will be best appreciated by those who are most familiar with the more abstruse subjects of analytical investigation. All the more complicated formulæ are however expressed according to the proposed system in the form of *notes*, as presenting the best opportunity of comparison with the more ordinary modes of expression in the text: and in some cases, this method has afforded a means of perspicuously indicating expansions so complicated as to be beyond the scope of ordinary symbols.

The principle of the proposed system is this: "if a_m is any function of m, and if A represents that operation by which any quantities are combined in a given manner, then the symbol $\overset{n}{\mathrm{A}}_m(a_m)$ may be used to represent the result of such combination applied to n functions, of which the m^{th} is a_m."

"Let then the letter P be taken as an abbreviation of the word *product*, and $\overset{n}{\mathrm{P}}_m(a_m)$ will denote a product of n factors of which the m^{th} is a_m."

The most common form in which factorials occur is that of an arithmetical series. As facility of writing is a point by no means unworthy of attention, it may be convenient to adopt an arbitrary symbol for factorials of this kind: that proposed by Mr. Jarrett appears, for clearness and uniformity, to be far superior to the symbols devised for the same purpose by Vandermonde, Kramp, and Arbogast. A product of m factors, of which the first is n, and the common difference $\pm r$, may be denoted by $\underline{|n}_{\,m,\pm r}$. In the particular case in which $r = -1$, this indication may be omitted, the symbol will then be $\underline{|n}_{\,m}$; and if in this case $m = n$, the index subscript may be omitted: thus

$$\underline{|n}_{\,m,\pm r} = n(n \pm r)(n \pm 2r)\ldots(n \pm \overline{m-1}.r),$$

$$\underline{|n}_{\,m} = n(n-1)(n-2)\ldots(n-m+1),$$

$$\underline{|n} = n(n-1)\ldots 3.2.1.$$

In the developments involving d and D, these functions usually appear under the form $\frac{d^m}{1.2\ldots m}$, in these cases the adoption of d^m: instead of $\frac{d^m}{\underline{|m}}$ will afford an additional facility, thus

$$D_x^m{:}u = \frac{D_x^m u}{1.2\ldots m}, \qquad d_x^m{:}d_y^n{:}u = \frac{d_x^m d_y^n u}{1.2\ldots m.1.2\ldots n}.$$

"The letter S may be taken as an abbreviation of the word *sum*, thus $\overset{n}{S}_m a_m$ will represent the sum of n terms of a series of which the m^{th} is a_m. It follows from this that the symbol $\overset{r}{S}_m \overset{s}{S}_n a_{m,n}$ will correctly represent the sum of a series consisting of r terms, of which the m^{th} term is the series $\overset{s}{S}_n a_{m,n}$; and the same notation may be extended to any number of symbols of summation."

A method analogous to the preceding has been devised by Mr. Jarrett for expressing by means of brackets the relation that exists between the different parts of a formula, when they are not connected either by addition or multiplication. An index is placed over the first bracket to denote the number of parts of which an expression consists, and a second bracket is so placed as to exhibit the connexion between two consecutive parts: thus

$$\underset{r}{\overset{n}{\Big\{}}\frac{a_r}{b_r+\underset{r+1}{\{}} = \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\dots\frac{a_n}{b_n};$$

$$\underset{r}{\overset{n}{\{}} a_r + b_r \underset{r+1}{\{} \dots \underset{n+1}{\{} c$$

$$= \underset{1}{\{} a_1 + b_1 \underset{2}{\{} a_2 + b_2 \underset{3}{\{} \dots \underset{n+1}{\{} c \underset{n+1}{\}} \dots \underset{1}{\}},$$

$$= \underset{1}{\{} a_1 + \underset{2}{\{} b_1 . a_2 + \underset{3}{\{} b_1 . b_2 . a_3 + \underset{4}{\{} \dots + \underset{n}{\{} b_1 . b_2 \dots b_{n-1} . a_n + \underset{n+1}{\{} b_1 . b_2 \dots b_n . c \underset{n+1}{\}} \dots \underset{1}{\}},$$

$$= a_1 + b_1 . a_2 + b_1 . b_2 . a_3 + \dots + b_1 . b_2 \dots b_{n-1} . a_n + b_1 . b_2 \dots b_n . c,$$

$$= \overset{n}{S}_m a_m . \overset{m-1}{P}_r (b_r) + c . \overset{m}{P}_r (b_r).$$

The brackets may be rejected after the expansion, whenever they are found not to possess any analytical signification; as in the preceding example.

In many investigations, particularly those connected with the theory of equations, it will be found convenient to denote by a symbol the sum of all the combinations of n quantities taken m at a time. Let therefore the letter C be taken as an abbre-

(6) If r is independent of n, and s of m, then

$$\overset{r}{\mathrm{S}}_m \overset{s}{\mathrm{S}}_n a_{m,n} = \overset{s}{\mathrm{S}}_n \overset{r}{\mathrm{S}}_m a_{m,n}.$$

(7) $\overset{\infty}{\mathrm{S}}_m \overset{\infty}{\mathrm{S}}_n a_{m,n} = \overset{\infty}{\mathrm{S}}_m \overset{m}{\mathrm{S}}_n a_{m-n+1,n}.$

(8) $\overset{\infty}{\mathrm{S}}_{m_1} \overset{\infty}{\mathrm{S}}_{m_2} \ldots \overset{\infty}{\mathrm{S}}_{m_s} \phi(m_1, m_2, \ldots m_s)$

$$= \overset{\infty}{\mathrm{S}}_{m_1} \overset{m_1}{\mathrm{S}}_{m_2} \ldots \overset{m_{s-1}}{\mathrm{S}}_{m_s} \phi(m_1 - m_2 + 1, \ldots m_{s-1} - m_s + 1, m_s).$$

(9) If $a_{m+1} - a_m = b_m$, then

$$a_{n+1} = a_1 + \overset{n}{\mathrm{S}}_m b_m.$$

(10) If $a_{m+1} + a_m = b_m$, then

$$(-1)^{n+1} a_{n+1} = -a_1 + \overset{n}{\mathrm{S}}_m (-1)^{m-1} . b_m.$$

(11) $\overset{m}{\mathrm{S}}_n \overset{n}{\mathrm{S}}_r a_{n,r} = \overset{m}{\mathrm{S}}_n \overset{m-n+1}{\mathrm{S}}_r a_{n+r-1,r}.$

(12) $\ldots\ldots = \overset{m}{\mathrm{S}}_n \overset{m-n+1}{\mathrm{S}}_r a_{n+r-1,n}.$

(13) $\overset{2m}{\mathrm{S}}_n \overset{n}{\mathrm{S}}_r a_{n,r} = \overset{m}{\mathrm{S}}_n \overset{n}{\mathrm{S}}_r (a_{2n-r,r} + a_{2n-r+1,r}).$

(14) $\overset{2m-1}{\mathrm{S}}_n \overset{n}{\mathrm{S}}_r a_{n,r} = \overset{m-1}{\mathrm{S}}_n \overset{n}{\mathrm{S}}_r (a_{2n-r,r} + a_{2n-r+1,r}) + \overset{2m-1}{\mathrm{S}}_r a_{2m-1,r}.$

(15) $\overset{\infty}{\mathrm{S}}_m \overset{m}{\mathrm{S}}_n a_{m,n} . b_n = \overset{\infty}{\mathrm{S}}_n b_n . \overset{\infty}{\mathrm{S}}_m a_{m+n-1,n}.$

(16) $(\overset{\infty}{\mathrm{S}}_m a_{m-1} . x^{m-1})(\overset{\infty}{\mathrm{S}}_n b_{n-1} . x^{n-1}) = \overset{\infty}{\mathrm{S}}_m x^{m-1} . \overset{m}{\mathrm{S}}_n a_{m-n} . b_{n-1}.$

[3] (1) If b is independent of r, then

$$\overset{m,n}{\mathrm{C}}_r (a_r . b) = b^m . \overset{m,n}{\mathrm{C}}_r (a_r).$$

(2) $\overset{m,n}{\mathrm{C}}_r (a_r) = \overset{n}{\mathrm{P}}_r (a_r) . \overset{n-m,n}{\mathrm{C}}_r \left(\frac{1}{a_r}\right).$

[4] (1) $\overset{\overline{n-m+1,m}}{\mathrm{C}}_{r,s} (a_r . b_s) = a_{n+1} . \overset{\overline{n-m,m}}{\mathrm{C}}_{r,s} (a_r . b_s) + b_{n+1} . \overset{\overline{n-m+1,m-1}}{\mathrm{C}}_{r,s} (a_r . b_s).$

(2) $\overset{\overline{n-m,m}}{\mathrm{C}}_{r,s} (a_r . b_s) = \overset{n}{\mathrm{P}}_r (a_r) . \overset{m,n}{\mathrm{C}}_r \left(\frac{b_r}{a_r}\right).$

Of the particular theorems which will here be considered, the first, and probably the most important, is the binomial theorem; the true development of which as first given by Mr. Swinburne and the Rev. T. Tylecote,* and as more compendiously demonstrated by Mr. Jarrett,† will afford a solution of all those contradictory results which have insinuated themselves into modern analysis, in consequence of neglecting the remainders in the development of diverging series: to obviate which, some writers have proposed a distinction between the *mathematical* signification of the term *equality*, and that which is implied in the ordinary acceptation of the term.

$$x^n = 1 + \overset{t}{\dot{\mathrm{S}}}_s\, \underline{|n}_{\,s} \cdot \frac{(x-1)^s}{\underline{|s}} + (x-1)^{t+1} \cdot \overset{n-t}{\mathrm{S}}_s\, \underline{|n-s}_{\,t} \cdot \frac{x^{s-1}}{\underline{|t}},$$

n being a positive integer.

$$x^{-n} = 1 + \overset{t}{\dot{\mathrm{S}}}_s\, \underline{|-n}_{\,s} \cdot \frac{(x-1)^s}{\underline{|s}} + (1-x)^{t+1} \cdot \overset{n}{\dot{\mathrm{S}}}_s\, \underline{|n+t-s}_{\,t} \cdot \frac{x^{-s}}{\underline{|t}}.$$

$$\frac{1}{\underline{|r}} \cdot \underline{\left|\frac{m}{n}\right.}_{\,r+1} \cdot \overset{r}{\dot{\mathrm{S}}}_s \frac{(-1)^{s-1}}{\underline{|s-1}} \cdot \underline{|r}_{\,s-1} \cdot \underline{|n(r-s+1)}_{\,t+1} \cdot \left(\frac{m}{n} - (r-s+1)\right)^{-1}$$

$$- \underline{|m}_{\,t+1} = 0, \left\{\begin{smallmatrix} t=0 \\ t=r-1 \end{smallmatrix}\right\}; \qquad \text{and } n^{r+1} \cdot \underline{\left|\frac{m}{n}\right.}_{\,r+1} = \underline{|m}_{\,r+1} -$$

$$\frac{1}{\underline{|r}} \cdot \underline{\left|\frac{m}{n}\right.}_{\,r+1} \cdot \overset{r}{\dot{\mathrm{S}}}_s \frac{(-1)^{s-1}}{\underline{|s-1}} \cdot \underline{|r}_{\,s-1} \cdot \underline{|n(r-s+1)}_{\,r+1} \cdot \left(\frac{m}{n} - (r-s+1)\right)^{-1}.$$

$$x^{\frac{m}{n}} = 1 + \overset{r}{\dot{\mathrm{S}}}_t\, \underline{\left|\frac{m}{n}\right.}_{\,t} \cdot \frac{(x-1)^t}{\underline{|t}} + \overset{\infty}{\dot{\mathrm{S}}}_t \frac{(x^{\frac{1}{n}}-1)^{r+t}}{\underline{|r+t}} \times$$

$$\left\{ \underline{|m}_{\,t+r} - \underline{\left|\frac{m}{n}\right.}_{\,r+1} \cdot \frac{1}{\underline{|r}} \cdot \overset{r}{\dot{\mathrm{S}}}_s \frac{(-1)^{s-1}}{\underline{|s-1}} \cdot \underline{|r}_{\,s-1} \cdot \underline{|n(r-s+1)}_{\,t+r} \cdot \left(\frac{m}{n} - (r-s+1)\right)^{-1} \right\}.$$

---

* *The true development of the Binomial Theorem*, Cambridge, 1827.

† *Cambridge Philosophical Transactions*, Vol. III.

The following theorem, which is nearly allied to the binomial, may easily be demonstrated by means of the theorems already given:

$$\frac{\underline{|a+b}_{n,r}}{\underline{|n}} = \overset{n+1}{\mathrm{S}}_m \frac{\underline{|a}_{n-m+1,r} \cdot \underline{|b}_{m-1,r}}{\underline{|n-m+1} \cdot \underline{|m-1}}.$$

If $r = 0$, this series is reduced to (β), p. 15. (*L. C. D.* 987.)

The polynomial theorem has been investigated by Arbogast, but with the assistance of a very cumbrous and obscure notation: the symbol devised by Mr. Jarrett, which has been adopted in the preceding pages may be thus explained. Let the character ϖ be taken as an abbreviation of the term *polynomial coefficient*; then the coefficient of x^m in the polynomial function

$$\phi(\overset{\infty}{\mathrm{S}}_m a_{m-1} \cdot x^{m-1})$$

may be represented by $\varpi_a^m \phi(a)$. The index subscript of ϖ is the letter according to the indices subscript of which the different powers of x ascend, and the quantity following the functional symbol is the term independent of x in the series

$$\overset{\infty}{\mathrm{S}}_m a_{m-1} \cdot x^{m-1}.$$

If the index subscript of ϖ is omitted, that letter is understood which immediately follows it, and if the function is a power of the polynomial, the parentheses including the first term of the polynomial may also be omitted: thus

$\varpi^m \phi(a)$ denotes the coefficient of x^m in $\overset{\infty}{\mathrm{S}}_m \phi_{m-1}(a) \cdot x^{m-1}$;

$\varpi^m a^n$ . . . . $\{\overset{\infty}{\mathrm{S}}_m a_{m-1} \cdot x^{m-1}\}^n$;

$\varpi^m a_r^n$ . . . . $\{\overset{\infty}{\mathrm{S}}_m a_{r+m-1} \cdot x^{m-1}\}^n$.

The demonstration of this theorem, for which the Author is indebted to Mr. Jarrett, may be given as an example of the facility with which very complicated operations may be conducted by means of the various symbols already explained:

[2] (2), $\overset{\infty}{\mathrm{S}}_m a_{m-1} \cdot x^{m-1} = a + x \cdot \overset{\infty}{\mathrm{S}}_m a_m \cdot x^{m-1} = a + x \cdot \overset{\infty}{\mathrm{S}}_r a_r \cdot x^{r-1}$;

(β) p. 15, $\frac{1}{\underline{|n}}\{\overset{\infty}{S}_m a_{m-1}.x^{m-1}\}^n$

$$=\overset{\infty}{S}_m \frac{a^{n-m+1}}{\underline{|n-m+1}}.\frac{x^{m-1}}{\underline{|m-1}}.\{\overset{\infty}{S}_r a_r.x^{r-1}\}^{m-1},$$

$$=\overset{\infty}{S}_m \frac{a^{n-m+1}}{\underline{|n-m+1}}.\frac{x^{m-1}}{\underline{|m-1}}.\overset{\infty}{S}_r x^{r-1}.\varpi^{r-1}a_1^{m-1},$$

[2] (7), $$=\overset{\infty}{S}_m x^{m-1}.\overset{m}{S}_r \frac{a^{n-m+r}}{\underline{|n-m+r}}.\frac{\varpi^{r-1}a_1^{m-r}}{\underline{|m-r}};$$

$$\therefore\quad \frac{\varpi^m a^n}{\underline{|n}}=\overset{m+1}{S}_r \frac{a^{n-m+r-1}}{\underline{|n-m+r-1}}.\frac{\varpi^{r-1}a_1^{m-r+1}}{\underline{|m-r+1}},$$

[2] (2), $$=\overset{m}{S}_r \frac{a^{n-m+r-1}}{\underline{|n-m+r-1}}.\frac{\varpi^{r-1}a_1^{m-r+1}}{\underline{|m-r+1}}+a^n.\varpi^m a_1^0,$$

$$=\overset{m}{S}_r \frac{a^{n-m+r-1}}{\underline{|n-m+r-1}}.\frac{\varpi^{r-1}a_1^{m-r+1}}{\underline{|m-r+1}},\ \text{since } \varpi^m a_1^0=0,$$

[2] (1), $$=\overset{m}{S}_r \frac{a^{n-r}}{\underline{|n-r}}.\frac{\varpi^{m-r}a_1^{r}}{\underline{|r}}.$$

The development of the first eight polynomial coefficients has already been given. (pp. 17—21.)

The value of the polynomial coefficients, in the particular case in which $n=-1$, may be obtained from the formula given in *Note* γ, p. 128; they may however be more readily deduced in the following manner: let $\varpi^m a^{-1}=b_m$, then

$$1=(\overset{\infty}{S}_m a_{m-1}.x^{m-1})(\overset{\infty}{S}_m b_{m-1}.x^{m-1}),$$

from which we obtain $b_m=-a^{-1}.\overset{m}{S}_r a_{m-r+1}.b_{r-1}$.

The actual values of some of the coefficients b, b_1, b_2, &c. may here be given, as the several terms of many series may be expressed by means of these quantities.

$b = a^{-1}$; $\quad b_1 = -a^{-2}.a_1$; $\quad b_2 = +a^{-3}.a_1^2 - a^{-2}.a_2$;

$b_3 = -a^{-2}.a_3 + 2a^{-3}.a_1.a_2 - a^{-4}a_1^3$;

$b_4 = +a^{-5}.a_1^4 - 3a^{-4}.a_1^2.a_2 + a^{-3}(2a_1.a_3 + a_2^2) - a^{-2}.a_4$;

$$b_5 = -a^{-6}.a_1^5 + 4a^{-5}.a_1^3.a_2 - 3a^{-4}(a_1^2.a_3 + a_1.a_2^2)$$
$$+ 2a^{-3}(a_1.a_4 + a_2.a_3) - a^{-2}.a_5;$$

$$b_6 = +a^{-7}.a_1^6 - 5a^{-6}.a_1^4.a_2 + a^{-5}(6a_1^2.a_2^2 + 4a_1^3.a_3)$$
$$-a^{-4}(6a_1.a_2.a_3 + 3a_1^2.a_4 + a_2^3) + a^{-3}(2a_1.a_5 + 2a_2.a_4 + a_3^2) - a^{-2}a^{6};$$

$$b_7 = -a^{-8}.a_1^7 + 6a^{-7}.a_1^5.a_2 - 5a^{-6}(2a_1^3.a_2^2 + a_1^4.a_3)$$
$$+ 4a^{-5}(3a_1^2.a_2.a_3 + a_1.a_2^3 + a_1^3.a_4) - 3a^{-4}(a_1.a_3^2 + a_2^2.a_3$$
$$+ 2a_1.a_2.a_4 + a_1^2.a_5) + 2a^{-3}(a_3.x_4 + a_2.a_5 + a_1.a_6) - a^{-2}.a_7.$$

The two following theorems may be placed here on account of their connexion with the preceding:

$$\left\{\overset{\infty}{S}_m \frac{x^{m-1}.\left\lfloor\underline{a}\right._{m-1,r}}{\left\lfloor\underline{m-1}\right.}\right\}\left\{\overset{\infty}{S}_n \frac{x^{n-1}.\left\lfloor\underline{b}\right._{n-1,r}}{\left\lfloor\underline{n-1}\right.}\right\}^{\pm 1} = \overset{\infty}{S}_m \frac{x^{m-1}.\left\lfloor\underline{a \pm b}\right._{m-1,r}}{\left\lfloor\underline{m-1}\right.};$$

$$\left\{\overset{\infty}{S}_m \frac{x^{m-1}.\left\lfloor\underline{a}\right._{m-1,r}}{\left\lfloor\underline{m-1}\right.}\right\}^{n} = \overset{\infty}{S}_m \frac{x^{m-1}.\left\lfloor\underline{na}\right._{m-1,r}}{\left\lfloor\underline{m-1}\right.}:$$

the first of these is Stainville's theorem.

Values of the logarithm of a factorial:

$$\log_{\varepsilon}\left\lfloor\underline{a}\right._{n,r} = n(\log_{\varepsilon} a - 1) + \left(n - \tfrac{1}{2} + \frac{a}{r}\right)\log_{\varepsilon}\left(1 + \frac{n}{a}r\right),$$
$$= n\log_{\varepsilon} a + \overset{\infty}{S}_m(-1)^{m-1}.\frac{r^m}{ma^m}.\overset{m-1}{S}_s s^m.$$

The three following theorems frequently occur in the theory of equations:

$$\overset{n}{P}_r(x + a_r) = \overset{n+1}{S}_m x^{m-1}.\overset{n-m+1,n}{C}_r(a_r);$$

$$\overset{n}{S}_m a_m^t = \overset{t-1}{S}_s(-1)^{s-1}.\overset{s,n}{C}_r(a_r).\overset{n}{S}_m a_m^{t-s} + (-1)^{t-1}.t.\overset{t,n}{C}_r(a_r);$$

$$\overset{\overline{m,n}}{C}_r(x+a_r)=\overset{m+1}{S}_s\,\frac{\lfloor\underline{n-s+1}_{\,n-m}}{\lfloor\underline{n-m}}\,.\,x^{m-s+1}.\overset{s-1,n}{C}_r(a_r).$$

$$\overset{n}{P}_r(a_r+b_r)=\overset{n+1}{S}_m\;\overset{\overline{n-m+1,m-1}}{C}_{r,s}(a_r.b_s)\,;$$

from this very useful theorem we may obtain the following: see *Trig.* (31.)

$$\cos\overset{n}{S}_r a_r=\overset{\infty}{S}_m(-1)^{m-1}.\overset{\overline{n-2m+2,2m-2}}{C}_{r,s}(\cos a_r.\sin a_s)\,;$$

$$\sin\overset{n}{S}_r a_r=\overset{\infty}{S}_m(-1)^{m-1}.\overset{\overline{n-2m+1,2m-1}}{C}_{r,s}(\cos a_r.\sin a_s)\,;$$

$$\tan\overset{n}{S}_r a_r=\frac{\sin\overset{n}{S}_r a_r}{\cos\overset{n}{S}_r a_r}=\frac{\overset{\infty}{S}_m(-1)^{m-1}.\overset{2m-1,n}{C}_r(\tan a_r)}{\overset{\infty}{S}_m(-1)^{m-1}.\overset{2m-2,n}{C}_r(\tan a_r)},\ \text{by [4] (2).}$$

The series (α, p. 294) may likewise be readily deduced from this theorem.

The theorem (*Note a*, p. 196) may be thus demonstrated:

$d_x^n{:}\phi(u)$ is the coefficient of $(Dx)^n$ in $E_x\phi(u)$,

and $\quad E_x\phi(u)=\phi(u+D_xu)$,

θ, p. 197, $\quad=\overset{\infty}{S}_n d_u^{n-1}{:}\phi(u).(D_xu)^{n-1}$,

. . . . . . $\quad=\overset{\infty}{S}_n d_u^{n-1}{:}\phi(u)\,\{\overset{\infty}{S}_m(Dx)^m.d_x^m{:}u\}^{n-1}$,

$$=\overset{\infty}{S}_n(Dx)^{n-1}.d_u^{n-1}{:}\phi(u)\,\{\overset{\infty}{S}_m(Dx)^{m-1}.d_x^m{:}u\}^{n-1},$$

$$=\overset{\infty}{S}_n(Dx)^{n-1}.d_u^{n-1}{:}\phi(u).\overset{\infty}{S}_m(Dx)^{m-1}.\varpi^{m-1}a^{n-1},$$

$$\{a_{r-1}=d_x^r{:}u\},$$

[2] (15), $\quad=\overset{\infty}{S}_n(Dx)^{n-1}.\overset{n}{S}_m d_u^{n-m}{:}\phi(u).\varpi^{m-1}a^{n-m}$;

$$\therefore\quad d_x^n{:}\phi(u)=\overset{n+1}{S}_m d_u^{n-m+1}{:}\phi(u).\varpi^{m-1}a^{n-m+1},$$

$$=\overset{n}{S}_m d_u^{n-m+1}{:}\phi(u).\varpi^{m-1}a^{n-m+1}+\phi(u).\varpi^n a^0,$$

$$= \overset{n}{\mathrm{S}}_m d_u^{n-m+1}{:}\phi(u) . \varpi^{m-1} a^{n-m+1}, \text{ since } \varpi^n a^0 = 0,$$

[2] (1),
$$= \overset{n}{\mathrm{S}}_m d_u^{m}{:}\phi(u) . \varpi^{n-m} a^{m},$$

$$= \overset{n}{\mathrm{S}}_m d_u^{m}\phi(u) . \frac{\varpi^{n-m} a^{m}}{\underline{|m}}.$$

The following theorems may be proved in a similar manner:

$$\phi(\overset{\infty}{\mathrm{S}}_m a_{m-1} . x^{m-1}) = \phi(a) + \overset{\infty}{\mathrm{S}}_n x^n . \overset{n}{\mathrm{S}}_m d_a^{n-m+1}\phi(a) . \frac{\varpi^{m-1} a_1^{n-m+1}}{\underline{|n-m+1}},$$

hence
$$\varpi_a^n \phi(a) = \overset{n}{\mathrm{S}}_m d_a^{n-m+1}\phi(a) . \frac{\varpi^{m-1} a_1^{n-m+1}}{\underline{|n-m+1}}.$$

$$d_x^n \phi(\overset{\infty}{\mathrm{S}}_m a_{m-1} . x^{m-1}) = \overset{\infty}{\mathrm{S}}_m x^{m-1} . \underline{|n+m-1}_{\,n} . \varpi_a^{n+m-1}\phi(a).$$

The first of these may be applied to the following examples:

$$\log_\epsilon(\overset{\infty}{\mathrm{S}}_m a_{m-1} . x^{m-1}) = \log_\epsilon a + \overset{\infty}{\mathrm{S}}_n x^n . \overset{n}{\mathrm{S}}_m (-1)^{n-m} . \frac{\varpi^{m-1} a_1^{n-m+1}}{(n-m+1) a^{n-m+1}};$$

$$\log_\epsilon^{-1}(\overset{\infty}{\mathrm{S}}_m a_{m-1} . x^{m-1}) = e^a \left\{ 1 + \overset{\infty}{\mathrm{S}}_n x^n . \overset{n}{\mathrm{S}}_m \frac{\varpi^{m-1} . a_1^{n-m+1}}{\underline{|n-m+1}} \right\}.$$

The following theorems will be found very useful in integration:

$$\int_x (u . v) = \overset{r}{\mathrm{S}}_m (-1)^{m-1} . d_x^{m-1} u . \int_x^m v + (-1)^r \int_x (d_x^r u . \int_x^r v).$$

If $a_n = a_{n-1} . b + a_{n-2} . c$, n being a positive integer, then

$$a_n = a_2 . \overset{\infty}{\mathrm{S}}_m \frac{\underline{|n-m-1}_{\,m-1}}{\underline{|m-1}} b^{n-2m} . c^{m-1} + a_1 . \overset{\infty}{\mathrm{S}}_m \frac{\underline{|n-m-2}_{\,m-1}}{\underline{|m-1}} b^{n-2m-1} . c^{m},$$

If $a_n = b_n + c_n . a_{n+t}$, where t is any quantity whatever, then

$$a_n = \overset{r}{\mathrm{S}}_m b_{n+(m-1)t} . \overset{m-1}{\mathrm{P}}_s (c_{n+(s-1)t}) + a_{n+rt} . \overset{r}{\mathrm{P}}_s (c_{n+(s-1)t}).$$

The following may be added to the theorems already given in finite differences.

$$\phi(x) = \overset{\infty}{S}_m \underline{\lfloor x}_{\,m-1,-Dx} \cdot \frac{D^{m-1}_{x=0} : \phi(x)}{(Dx)^{m-1}}.$$

$$\phi(\epsilon^x) = \overset{\infty}{S}_m \frac{x^{m-1}}{\underline{\lfloor m-1}} \cdot \phi(1+\Delta)0^{m-1}.$$

$$(\epsilon^x - 1)^n = \overset{\infty}{S}_m \frac{x^{n-m+1}}{\underline{\lfloor n-m+1}} \cdot \Delta^n 0^{n+m-1}.$$

$$(\epsilon^x + 1)^{-1} = \overset{\infty}{S}_m \frac{x^{m-1}}{\underline{\lfloor m-1}} \cdot \overset{\infty}{S}_n (-1)^{n-1} \cdot \frac{\Delta^{n-1} 0^{m-1}}{2^n}.$$

$$D_x^m \underline{\lfloor x}_{\,\pm n, Dx} = \underline{\lfloor \pm n}_{\,m} \cdot (Dx)^m \cdot \underline{\lfloor x + mDx}_{\,\pm n-m, Dx};$$

$$D_x^m \underline{\lfloor x}_{\,\pm n, -Dx} = \underline{\lfloor \pm n}_{\,m} \cdot (-Dx)^m \cdot \underline{\lfloor x - mDx}_{\,\pm n-m, -Dx}.$$

For the theorems which have here been given, as instances of the great power of this system of notation, the Author is indebted to the unpublished researches of Mr. Jarrett: many of these are entirely original, and the others appear under such an altered form, that a reference to the authors from whose works they have been taken would not at all assist the reader in deducing them.

In concluding this subject, a slight modification of the symbols P and C may be mentioned, which Mr. Jarrett has found to afford considerable facility in many important investigations: the symbols $\overset{n;t}{P}_r(a_r)$ and $\overset{m,n;t}{C}_r(a_r)$ express the values which $\overset{n}{P}_r(a_r)$ and $\overset{m,n}{C}_r(a_r)$ respectively assume, when a_t is omitted, wherever it is found to occur. This will be best understood by an example: given

$$b_{r-1} = \overset{n}{S}_m a_m^{r-1} . x_m, \left\{ {r=1 \atop r=n} \right\},$$

to find x_t. (*Cauchy*, *cours d'Analyse*, Pt. 1, p. 71.)

Multiply by $\overset{n-r,n;t}{\mathrm{C}}_s(-a_s)$, then

$$b_{r-1}\cdot\overset{n-r,n;t}{\mathrm{C}}_s(-a_s)=\overset{n}{\mathrm{S}}_m x_m\cdot a_m^{r-1}\cdot\overset{n-r,n;t}{\mathrm{C}}_s(-a_s),$$

$$\therefore\quad \overset{n}{\mathrm{S}}_r b_{r-1}\cdot\overset{n-r,n;t}{\mathrm{C}}_s(-a_s)=\overset{n}{\mathrm{S}}_m x_m\cdot\overset{n}{\mathrm{S}}_r a_m^{r-1}\cdot\overset{n-r,n;t}{\mathrm{C}}_s(-a_s),$$

$$=\overset{n}{\mathrm{S}}_m x_m\cdot\overset{n;t}{\mathrm{P}}_r(a_m-a_r).$$

But $\overset{n;t}{\mathrm{P}}_r(a_m-a_r)=0$, for every value from $m=1$, to $m=n$, except when $m=t$;

$$\therefore\quad \overset{n}{\mathrm{S}}_r b_{r-1}\cdot\overset{n-r,n;t}{\mathrm{C}}_s(-a_s)=x_t\cdot\overset{n;t}{\mathrm{P}}_r(a_t-a_r),$$

and

$$x_t=\frac{\overset{n}{\mathrm{S}}_r b_{r-1}\cdot\overset{n-r,n;t}{\mathrm{C}}_s(-a_s)}{\overset{n;t}{\mathrm{P}}_r(a_t-a_r)}.$$

If $b_{r-1}=b^{r-1}$, then

$$x_t=\frac{\overset{n}{\mathrm{S}}_r b^{r-1}\cdot\overset{n-r,n;t}{\mathrm{C}}_s(-a_s)}{\overset{n;t}{\mathrm{P}}_r(a_t-a_r)}=\overset{n;t}{\mathrm{P}}_r\left(\frac{b-a_r}{a_t-a_r}\right).$$